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OPTIMAL TACTICS IN A MULTISTRIKE

AIR CAMPAIGN (U)

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SUMMARY

A model of a tactical air campaign consisting of a finite number of counterair and countersurface missions is analyzed as a multimove continuous game. The optimal tactics are derived as a function of the attrition parameters. It is shown that both sides have optimal pure strategies — neither side ever needs to bluff although the campaign consists of simultaneous moves. The optimal tactics for one side are either (a) allocate all forces to countersurface missions during each strike, or (b) allocate all forces to counterair during some initial period and to countersurface thereafter. The optimal tactics for the other side are (a) or (b) or (c) a possible split of forces during the early phase of the campaign with the campaign ending as (b).



## OPTIMAL TACTICS IN A MULTISTRIKE AIR CAMPAIGN

Melvin Dresner

### 1. Introduction

In a previous memorandum [1] we derived a solution to the tactical air game as formulated by Fulkerson and Johnson [2] for certain classes of attrition parameters. It was shown that air campaigns having these attrition parameters have optimal pure strategies of an "all or nothing" type.

In this memorandum we solve this tactical air campaign for arbitrary attrition parameters. We show that, independent of the attrition parameters and initial conditions, both players have optimal pure strategies. Although every move by a player in this game is made simultaneously with his opponent, nevertheless, a player never needs to randomize. However, the optimal strategies are not always of an "all or nothing" type — they may involve division of forces.

The argument in [1] assumed the principle of optimality. This, in effect, imposed the condition that after a certain stage both players play optimally. However, by defining strategy as suggested by I. Glicksberg, we solve the tactical air game without using the principle of optimality.

### 2. The Tactical Air Campaign

We shall describe the game by describing the strategies and giving the payoff associated with pairs of strategies. For a further description of the game, see [1].

Let  $\Gamma_{n+1}$  be an air campaign in which Blue and Red each make  $n + 1$  strikes (moves), each strike by Blue being made at the same time as by Red. Let  $p_{n+1}$ ,  $q_{n+1}$  be the number of planes or resources possessed by Blue and Red, respectively, at the start of  $\Gamma_{n+1}$ .

Define

- a = fraction of Blue forces surviving antiaircraft fire, accidents, etc. between  $\Gamma_{n+1}$  and  $\Gamma_n$ .
- b = Red kill potential per plane sent against Blue's aircraft.
- c = fraction of Red forces surviving antiaircraft fire, accidents, etc. between  $\Gamma_{n+1}$  and  $\Gamma_n$ .
- d = Blue kill potential per plane sent against Red aircraft.
- $r_{n+1}$  = increase in Blue's forces between  $\Gamma_{n+1}$  and  $\Gamma_n$ . These forces are subject to attack but cannot be used for attack.
- $s_{n+1}$  = increase in Red's forces between  $\Gamma_{n+1}$  and  $\Gamma_n$ . They are subject to attack but cannot be used for attack.

A strike in  $\Gamma_{n+1}$  by Blue determines Red's resources  $q_n$  for  $\Gamma_n$ . Thus a strike by Blue is a choice  $q_n$  satisfying

$$(1) \quad \bar{q}_n = \max(0, cq_{n+1} + s_{n+1} - dp_{n+1}) \leq q_n \leq cq_{n+1} + s_{n+1} = \bar{\bar{q}}_n.$$

A strike in  $\Gamma_{n+1}$  by Red is a choice  $p_n$  such that

$$(2) \quad \overline{p}_n = \max(0, ap_{n+1} + r_{n+1} - bq_{n+1}) \leq p_n < ap_{n+1} + r_{n+1} = \overline{\overline{p}}_n$$

A strike in  $\Gamma_{n+1}$  by Blue is also an allocation of part of  $p_{n+1}$  against Red's airfields with the remainder dispatched on ground-support. Now if in  $\Gamma_{n+1}$  Blue chooses  $q_n$ , then Red loses  $(q_{n+1} + s_{n+1} - q_n)$  planes during this strike. Of this total  $(1-c)q_{n+1}$  planes will be lost to antiaircraft fire, and the remainder,  $(cq_{n+1} + s_{n+1} - q_n)$ , will be lost to airfield strikes by Blue. This implies that Blue allocates  $\frac{1}{d}(cq_{n+1} + s_{n+1} - q_n)$  planes against Red's airfields and the remainder,  $p_{n+1} - \frac{1}{d}(cq_{n+1} + s_{n+1} - q_n)$  planes, on ground support. Similarly, a choice of  $p_n$  in  $\Gamma_{n+1}$  by Red implies that Red dispatches  $q_{n+1} - \frac{1}{b}(ap_{n+1} + r_{n+1} - p_n)$  planes on ground support.

The payoff, which is measured by the total of Blue's superiority in ground-support during the campaign, is given by

$$\begin{aligned}
 (3) \quad K_{n+1}(\sigma, \tau) &= \sum_{i=0}^n \left[ p_{i+1} - \frac{cq_{i+1} + s_{i+1} - q_i}{d} - \left( q_{i+1} - \frac{ap_{i+1} + r_{i+1} - p_i}{b} \right) \right] \\
 &= \sum_{i=0}^n \left[ \frac{a+b}{ab} \overline{p}_i - \frac{c+d}{cd} \overline{q}_i - \frac{r_{i+1}}{a} + \frac{s_{i+1}}{c} + \frac{q_i}{d} - \frac{p_i}{b} \right].
 \end{aligned}$$

where  $\sigma$  and  $\tau$  are strategies of Blue and Red. Letting  $L_1(p_1, q_1)$  represent the quantity in brackets of (3), we can write the payoff as

$$(4) \quad K_{n+1}(\sigma, \tau) = \sum_{i=0}^n L_1(p_i, q_i).$$

Let  $\Sigma_n(p_n, q_n)$  be the set of pure strategies of Blue in the game  $\Gamma_n$ . Then we define a pure strategy in  $\Gamma_{n+1}$  for Blue as a vector

$$(5) \quad \sigma_{n+1}(p_{n+1}, q_{n+1}) = \| q_n, \phi_n \|$$

where

$$\bar{q}_n = \max(0, cq_{n+1} + s_{n+1} - dp_{n+1}) \leq q_n \leq cq_{n+1} + s_{n+1} = \bar{\bar{q}}_n$$

and  $\phi_n$  is a function which maps  $p_n, q_n$  into  $\Sigma_n(p_n, q_n)$ . We have, in particular,  $\sigma_1(p_1, q_1) = q_0$  where  $\bar{q}_0 \leq q_0 \leq \bar{\bar{q}}_0$ .

Similarly, if  $T_n(p_n, q_n)$  is the set of pure strategies in  $\Gamma_n$  for Red, then in  $\Gamma_{n+1}$  a pure strategy for Red is a vector

$$(6) \quad \tau_{n+1}(p_{n+1}, q_{n+1}) = \| p_n, \psi_n \|$$

where

$$\bar{p}_n = \max(0, ap_{n+1} + r_{n+1} - bq_{n+1}) \leq p_n \leq ap_{n+1} + r_{n+1} = \bar{\bar{p}}_n$$

and  $\psi_n$  is a function which maps  $p_n, q_n$  into  $T_n(p_n, q_n)$ .

From (4) we have

$$(7) \quad K_{n+1}(\sigma_{n+1}, \tau_{n+1}) = L_n(p_n, q_n) + K_n(\sigma_n, \tau_n),$$



where  $\sigma_n = \phi_n(p_n, q_n)$ ,  $\tau_n = \psi_n(p_n, q_n)$  and  $K_0 = 0$ .

### 3. Existence of Optimal Pure Strategies

We shall obtain a sufficient condition for the existence of optimal pure strategies. Suppose that, for any  $p_n, q_n$ , the game  $\Gamma_n$  has a value  $v_n(p_n, q_n)$  which is continuous in  $p_n, q_n$ , and suppose that Blue and Red have optimal pure strategies  $\sigma_n^*(p_n, q_n)$  and  $\tau_n^*(p_n, q_n)$ , respectively. Then in  $\Gamma_{n+1}$  we have, for any strategy  $\tau_{n+1}$  of Red and the strategy  $\| q_n, \phi_n^* \|$  of Blue, where  $\phi_n^*$  maps  $p_n, q_n$  onto  $\sigma_n^*(p_n, q_n)$ , that

$$\begin{aligned}
 (8) \quad K_{n+1}(\| q_n, \phi_n^* \|, \tau_{n+1}) &= L_n(p_n, q_n) + K_n(\sigma_n^*, \tau_n) \\
 &\geq L_n(p_n, q_n) + v_n(p_n, q_n) \\
 &\geq \min_{p_n} [L_n(p_n, q_n) + v_n(p_n, q_n)].
 \end{aligned}$$

The right side of (8) is a continuous function of  $q_n$ . Hence there exists some  $q_n^*$  which maximizes the right side, or

$$(9) \quad K_{n+1}(\| q_n^*, \phi_n^* \|, \tau_{n+1}) \geq \max_{q_n} \min_{p_n} [L_n(p_n, q_n) + v_n(p_n, q_n)]$$

for all  $\tau_{n+1}$ . Similarly, we have

$$(10) \quad K_{n+1}(\sigma_{n+1}, \| p_n^*, \psi_n^* \|) \leq \min_{p_n} \max_{q_n} [L_n(p_n, q_n) + v_n(p_n, q_n)]$$

for all  $\sigma_{n+1}$ .

As in [1], let us define

$$(11) \quad M_{n+1}(q_n, p_n) = L_n(p_n, q_n) + v_n(p_n, q_n) .$$

Now suppose that

$$\min_{q_n} \max_{p_n} [L_n(p_n, q_n) + v_n(p_n, q_n)] = \min_{p_n} \max_{q_n} [L_n(p_n, q_n) + v_n(p_n, q_n)] ,$$

or that  $M_{n+1}(q_n, p_n)$  has a saddle-point at  $q_n^*, p_n^*$ . Then from (9) and (10) we have

$$K_{n+1}(\sigma_{n+1}^*, \tau_{n+1}^*) \geq K_{n+1}(\sigma_{n+1}^*, \tau_{n+1}^*) \geq K_{n+1}(\sigma_{n+1}, \tau_{n+1}^*) ,$$

or  $\Gamma_{n+1}$  has a saddle-point at  $\sigma_{n+1}^*, \tau_{n+1}^*$ . Further,

$$(12) \quad v_{n+1}(p_{n+1}, q_{n+1}) = M_{n+1}(q_n^*, p_n^*)$$

which, it can be easily verified, is a continuous function of  $p_{n+1}, q_{n+1}$ . We have thus proved the following theorem.

Theorem 1. If  $\sigma_n^*, \tau_n^*$  are optimal pure strategies of  $\Gamma_n$  where  $v_n(p_n, q_n)$  is a continuous function of  $p_n, q_n$ , and if  $M_{n+1}(q_n, p_n)$  has a saddle-point at  $q_n^*, p_n^*$ , then  $\|\sigma_n^*, \tau_n^*\|, \|p_n^*, q_n^*\|$  are optimal pure strategies of  $\Gamma_{n+1}$  and  $v_{n+1}(p_{n+1}, q_{n+1}) = M_{n+1}(q_n^*, p_n^*)$ , which is also continuous.

The existence of optimal pure strategies in  $\Gamma_{n+1}$  therefore depends on the existence of a saddle-point of the function  $M_{n+1}(q_n, p_n)$ . We shall show that, for all  $n$ ,  $M_{n+1}(q_n, p_n)$  has a saddle point. We shall also compute the saddle-point whenever its location is independent of the initial resources  $p_{n+1}$ ,  $q_{n+1}$ .

#### 4. Campaigns of Short Duration

It is easily verified that  $M_1(q_0, p_0)$  has a saddle-point at  $\bar{q}_0$ ,  $\bar{p}_0$  and  $v_1(p_1, q_1) = p_1 - q_1$ .

Let  $f$  be the largest integer for which  $\frac{1}{b} - \frac{1-a^f}{1-a} \geq 0$ , and let  $g$  be the largest integer for which  $\frac{1}{d} - \frac{1-c^g}{1-c} \geq 0$ . Suppose that  $f < g$ . It can be shown by induction (the proof is given in [1]) that if  $1 \leq n \leq f$  then  $M_{n+1}(q_n, p_n)$  has a saddle-point at  $\bar{q}_n$ ,  $\bar{p}_n$  and

$$(13) \quad v_{n+1}(p_{n+1}, q_{n+1}) = \frac{1-a^{n+1}}{a(1-a)} \bar{p}_n - \frac{1-c^{n+1}}{c(1-c)} \bar{q}_n + \epsilon_{n+1}$$

where  $\epsilon_{n+1}$  is independent of  $p_{n+1}$  and  $q_{n+1}$ . Combining this result with Theorem 1, we obtain

Theorem 2. If  $n \leq f + 1$ , then  $\Gamma_n$  has optimal pure strategies of the form  $\| \bar{q}_{n-1}, \phi_{n-1}^* \|, \| \bar{p}_{n-1}, \psi_{n-1}^* \|$ .

If  $g < f$ , then in Theorem 2,  $f$  is replaced by  $g$ . It follows that during the last  $[1 + \min(f, g)]$  strikes of every campaign, both sides concentrate on ground-support. This result is independent of the attrition parameters.

5. Campaigns of Duration  $\leq t + 1$

Letting  $n = f$  in (13) and letting  $n = f + 1$  in (11) we have

$$(14) \quad M_{f+2}(q_{f+1}, p_{f+1}) = \frac{a+b}{ab} \bar{p}_{f+1} - \frac{c+d}{cd} \bar{q}_{f+1} + \epsilon_{f+2} \\
 + q_{f+1} \left( \frac{1}{d} - \frac{1-c^{f+1}}{1-c} \right) - p_{f+1} \left( \frac{1}{b} - \frac{1-a^{f+1}}{1-a} \right).$$

Since  $f < g$ , it follows that  $\frac{1}{d} - \frac{1-c^{f+1}}{1-c} > 0$  and  $\frac{1}{b} - \frac{1-a^{f+1}}{1-a} < 0$ .

Therefore  $M_{f+2}(q_{f+1}, p_{f+1})$  has a saddle-point at  $\bar{q}_{f+1}, \bar{p}_{f+1}$ .

Now let  $t$  be the largest value of  $n$  for which  $M_n(q_{n-1}, p_{n-1})$  has a saddle-point at  $\bar{q}_{n-1}, \bar{p}_{n-1}$  — i.e., for each  $n$ , where  $f + 2 \leq n \leq t$ ,

$\Gamma_n$  has optimal pure strategies of the form  $\| \bar{q}_{n-1}, \phi_{n-1}^* \|$ ,

$\| \bar{p}_{n-1}, \psi_{n-1}^* \|$  and  $M_{t+1}(q_t, p_t)$  does not have a saddle-point at  $\bar{q}_t, \bar{p}_t$ .

It can be shown (the complete proof is given in [1]) that if

$a + b > 1$ , then, for  $f + 2 \leq n \leq t$ ,

$$(15) \quad v_n(p_n, q_n) = \frac{a+b}{ab} \bar{p}_{n-1} - \beta_{n-1} \bar{q}_{n-1} + \gamma_{n-1} \bar{p}_{n-1} + \epsilon_n$$

where  $\beta_{n-1}, \gamma_{n-1} \geq 0$ . We have that  $t \leq g + 1$  but we cannot evaluate  $t$  as a function of the attrition parameters. The value of  $t$  probably depends on the initial resources as well as the attrition parameters.

Theorem 3. If  $a + b > 1$ , then for all  $n$ , such that  $f + 2 \leq n \leq t$ ,  $\Gamma_n$  has optimal pure strategies of the form  $\| \bar{q}_{n-1}, \phi_{n-1}^* \|, \| \bar{p}_{n-1}, \psi_{n-1}^* \|$ .

Having solved  $\Gamma_n$  for all  $n \leq t$ , we can now solve  $\Gamma_{t+1}$ .  
 Letting  $n = t$  in (15) we verify that  $M_{t+1}(q_t, p_t)$  has a saddle-  
 point at  $\bar{q}_t, \bar{p}_t$  and

$$(16) \quad v_{t+1}(p_{t+1}, q_{t+1}) = \frac{a+b}{ab} \bar{p}_t - \frac{c+d}{cd} \bar{q}_t + \gamma_t \bar{p}_t - \delta_t \bar{q}_t + \epsilon_{t+1}$$

where  $\gamma_t, \delta_t \geq 0$ . Hence

Theorem 4. If  $a + b > 1$ , then there exists an integer  $t$ ,  
where  $f + 2 \leq t \leq g + 1$  such that  $\Gamma_{t+1}$  has optimal pure strategies  
 $\|\bar{q}_t, \phi_t^*\|, \|\bar{p}_t, \psi_t^*\|$ .

6. Campaigns of Duration  $\geq t + 1$

We can now analyze games  $\Gamma_n$  where  $n \geq t + 1$ . We shall show  
 by induction that for these games

$$(17) \quad v_n(p_n, q_n) = \frac{a+b}{ab} \bar{p}_{n-1} - \alpha_{n-1} \bar{q}_{n-1} + \gamma_{n-1} \bar{p}_{n-1} - \delta_{n-1} \bar{q}_{n-1} + \epsilon_n$$

where  $\alpha_{n-1}, \gamma_{n-1}, \delta_{n-1} \geq 0$ .

Define  $\mu_n = ap_n + r_n - bq_n$  and  $\lambda_n = cq_n + s_n - dp_n$ . Assume  
 that (17) holds for some  $n \geq t + 1$ , then we have

$$(18) \quad M_{n+1}(q_n, p_n) = \frac{a+b}{ab} \bar{p}_n - \frac{c+d}{cd} \bar{q}_n + \epsilon_{n+1}$$

$$+ \begin{cases} q_n \left[ \frac{1}{d} - c\alpha_{n-1} \right] + p_n \left[ \frac{a+b-1}{b} \right] & \text{if } 0 \geq \mu_n, 0 \geq \lambda_n \\ q_n \left[ \frac{1}{d} - c\alpha_{n-1} - c\delta_{n-1} \right] + p_n \left[ \frac{a+b-1}{b} + d\delta_{n-1} \right] & \text{if } 0 \geq \mu_n, 0 \leq \lambda_n \\ q_n \left[ \frac{1}{d} - c\alpha_{n-1} - b\gamma_{n-1} \right] + p_n \left[ \frac{a+b-1}{b} + a\gamma_{n-1} \right] & \text{if } 0 \leq \mu_n, 0 \geq \lambda_n \\ q_n \left[ \frac{1}{d} - c\alpha_{n-1} - c\delta_{n-1} - b\gamma_{n-1} \right] + p_n \left[ \frac{a+b-1}{b} + a\gamma_{n-1} + d\delta_{n-1} \right] & \text{if } 0 \leq \mu_n, 0 \leq \lambda_n. \end{cases}$$

Assuming that  $a + b > 1$ , we may write (18) as

$$(19) \quad M_{n+1}(q_n, p_n) = \frac{a+b}{ab} \bar{p}_n - \frac{c+d}{cd} \bar{q}_n + \epsilon_{n+1}$$

$$+ \left( \frac{1}{d} - A_1 \right) q_n + B_1 p_n$$

where  $A_1 \geq 0$ ,  $B_1 \geq 0$  each take on four values defined in (18).

Since  $B_1 \geq 0$ , then for any  $q_n$ ,

$$M_{n+1}(q_n, \bar{p}_n) \leq M_{n+1}(q_n, p_n)$$

for all  $p_n$ . In particular, let  $q_n^*$  be that  $q_n$  which maximizes  $M_{n+1}(q_n, \bar{p}_n)$ , then we have

$$M_{n+1}(q_n, \bar{p}_n) \leq M_{n+1}(q_n^*, \bar{p}_n) \leq M_{n+1}(q_n^*, p_n) .$$

Hence  $M_{n+1}(q_n, p_n)$  has a saddle-point at  $q_n^*, \bar{p}_n$  where  $\bar{q}_n \leq q_n^* \leq \bar{q}_n$  .  
 (The existence of the saddle-point also follows from the fact that if one player has a dominating strategy then the game has a saddle-point.) Therefore

$$(20) \quad v_{n+1}(p_{n+1}, q_{n+1}) = \frac{a+b}{ab} \bar{p}_n - \frac{c+d}{cd} \bar{q}_n + (\frac{1}{d} - A_K) q_n^* + B_K \bar{p}_n + \epsilon_{n+1}$$

where  $A_K, B_K$  is one of the four pairs defined in (18).

In order to evaluate (20) further, we shall derive some properties of  $q_n^*$ , where  $n \geq t + 1$ . From (16) we note that  $\alpha_t = \frac{c+d}{cd}$  . Let us add to the induction hypothesis about  $v_n(p_n, q_n)$  as given by (17) that  $\alpha_{n-1} = \frac{c+d}{cd}$ , for  $n \geq t + 1$ . Then if we assume that  $c + d > 1$ , it follows that in (19),  $\frac{1}{d} - A_i \leq 0$  for each of the four values of  $A_i$ . Hence  $q_n^* = \bar{q}_n$  and

$$(21) \quad v_{n+1}(p_{n+1}, q_{n+1}) = \frac{a+b}{ab} \bar{p}_n - \frac{c+d}{cd} \bar{q}_n + \gamma_n \bar{p}_n - \delta_n \bar{q}_n + \epsilon_{n+1}$$

which agrees with (19) for  $n + 1$ . Hence in the case that  $c + d > 1$ , we have that  $M_{n+1}(q_n, p_n)$  has a saddle-point at  $\bar{q}_n, \bar{p}_n$ , for all  $n \geq t + 1$ .

This combined with Theorem 1 yields

Theorem 4. If  $a + b > 1, c + d > 1$ , then for all  $n$  such that  $n \geq t + 1$ ,  $\Gamma_n$  has optimal pure strategies of the form  $\| \bar{q}_{n-1}, \phi_{n-1}^* \|$ ,  $\| \bar{p}_{n-1}, \psi_{n-1}^* \|$  .

7. The Case  $a + b > 1, c + d < 1$

From the assumption that (17) holds for some  $n \geq t + 1$ , and that  $a + b > 1$ , it followed that  $M_{n+1}(q_n, p_n)$  had a saddle-point at  $q_n^*, \bar{p}_n$ . We still need to show that (17) holds in general for  $n + 1$ .

Let us set

$$(22) \quad q_n^* = z\bar{q}_n + (1-z)\bar{q}_n$$

where  $0 \leq z \leq 1$ . Suppose  $z = 1$ , or  $q_n^* = \bar{q}_n$ . Then it follows that in (20) we must have  $\frac{1}{d} - A_K \geq 0$ , and hence (20) reduces to

$$(23) \quad v_{n+1}(p_{n+1}, q_{n+1}) = \frac{a+b}{ab} \bar{p}_n - \left(\frac{1}{c} + A_K\right) \bar{q}_n + B_K \bar{p}_n + \epsilon_{n+1}$$

which agrees with (17) for  $n + 1$ .

Suppose  $z = 0$ , or  $q_n^* = \bar{q}_n$ . Since  $\bar{q}_n$  is optimal, then in (20) we must have  $\frac{1}{d} - A_K \leq 0$ , and (20) simplifies to

$$(24) \quad v_{n+1}(p_{n+1}, q_{n+1}) = \frac{a+b}{ab} \bar{p}_n - \frac{c+d}{cd} \bar{q}_n - \left(A - \frac{1}{d}\right) \bar{q}_n + B \bar{p}_n + \epsilon_{n+1}$$

which again agrees with (17) for  $n + 1$ .

Finally, suppose  $0 < z < 1$ , then either  $a\bar{p}_n + r_n - bq_n^* = 0$  or  $cq_n^* + sn - d\bar{p}_n = 0$ . For, if both were different from zero, then  $q_n^*, \bar{p}_n$  would not be a saddle-point of  $M_{n+1}(q_n, p_n)$ . Suppose



then  $s\bar{p}_n + r_n - bq_n^* = 0$  and  $cq_n^* + s_n - d\bar{p}_n < 0$ . It follows from the optimality of  $q_n^*$  that in (18) we must have  $\frac{1}{d} - c\alpha_{n-1} > 0$  and  $\frac{1}{d} - c\alpha_{n-1} - b\gamma_{n-1} < 0$ . Hence (20) reduces to

$$(25) \quad v_{n+1}(p_{n+1}, q_{n+1}) = \frac{s+b}{ab} \bar{p}_n - \left( \frac{1}{c} + \frac{1-z}{d} + z c \alpha_{n-1} + z b \gamma_{n-1} \right) \bar{q}_n \\
 - (1-z) \left( -\frac{1}{d} + c \alpha_{n-1} + b \gamma_{n-1} \right) \bar{q}_n + B \bar{p}_n + \epsilon_{n+1}$$

which agrees with (17) for  $n + 1$ . The argument for the remaining three cases is the same as the preceding. Hence the induction is complete.

For  $n \geq t + 2$ , we can construct examples, by altering the initial conditions and attrition parameters, in which  $q_n^*$  takes on any specified value between  $\bar{q}_n$  and  $\bar{q}_n$ .

Theorem 5. If  $a + b > 1$ ,  $c + d < 1$ , then for all  $n$ , such that  $n \geq t + 2$ ,  $\Gamma_n$  has optimal pure strategies of the form  $\| q_{n-1}^*, \phi_{n-1}^* \|, \| \bar{p}_{n-1}^*, \psi_{n-1}^* \|$ .

## 8. Application to Air Campaign

In the preceding discussion we have assumed that  $f < g$ . The argument is the same if  $g < f$  and the form of the solution is essentially the same, but the players are reversed.

Finally, we need to consider the case  $a + b < 1$ ,  $c + d < 1$ . For such attrition parameters it was shown in [1] that, for all  $n$ ,  $M_{n+1}(q_n, p_n)$  has a saddle-point at  $\bar{q}_n, \bar{p}_n$ . Hence the optimal strategies of  $\Gamma_n$  are of the form  $\| \bar{q}_{n-1}, \phi_{n-1}^* \|, \| \bar{p}_{n-1}, \psi_{n-1}^* \|$ .

In terms of the air campaign  $\Gamma_n$ , a choice  $q_{n-1} = \bar{q}_{n-1} (p_{n-1} = \bar{p}_{n-1})$  implies that on the n-th strike, measured from the end of the campaign, Blue (Red) sends his entire force on ground-support missions. Let us designate this by G. A choice  $q_{n-1} = \bar{q}_{n-1} (p_{n-1} = \bar{p}_{n-1})$  implies that Blue (Red) concentrates his forces on counterair missions. Designate this by A. Let (A,G) designate that a player does one of three things — A or G or splits his forces between ground-support and counterair missions. The following table summarizes the optimal strategies in an air campaign  $\Gamma_n$ .

Optimal Allocation of Forces  
 On n-th Strike of Air Campaign  
 (counting from end of campaign)

Attrition Parameters	$1 \leq n \leq \min(f+1, g+1)$	$\min(f+2, g+2) \leq n \leq t$	$n = t+1$	$t+2 \leq n \leq N$
$a+b > 1, c+d > 1, f < g$				
Blue	G	G	A	A
Red	G	A	A	A
$a+b > 1, c+d > 1, g < f$				
Blue	G	A	A	A
Red	G	G	A	A
$a+b > 1, c+d < 1$				
Blue	G	G	A	(A, G)
Red	G	A	A	A
$a+b < 1, c+d > 1$				
Blue	G	A	A	A
Red	G	G	A	(A, G)
$a+b < 1, c+d < 1$				
Blue	G	G	G	G
Red	G	G	G	G
$f = g$				
Blue	G	A	A	A
Red	G	A	A	A

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