U. S. AIR FORCE

PROJECT RAND

RESEARCH MEMORANDUM

AN APPROXIMATE SOLUTION
FOR A COVERAGE PROBLEM

H. H. Germond

RM-134

11 April 1949

Assigned to ______________________________

This is a working paper. It may be expanded, modified, or withdrawn at any time. The views, conclusions, and recommendations expressed herein do not necessarily reflect the official views or policies of the United States Air Force.

The RAND Corporation
1700 MAIN ST. • SANTA MONICA • CALIFORNIA

AN APPROXIMATE SOLUTION FOR A COVERAGE PROBLEM

H. H. Germond

Summary

This paper presents a method for making an approximate estimate of the total coverage which may be expected when a number of patterns are individually aimed at the same point.

The coverage expected from a single pattern, when the target is so large that there is no chance of the pattern falling outside the target area, is of course equal to the area of the pattern itself. The total coverage to be expected when two patterns are used is subject to exact calculation. The separation, x, between centers of the patterns which would give the same coverage is computed. The first two patterns are replaced, for purposes of computation, by a single circle whose area equals the expected coverage of the first two patterns. It is assumed that the expected coverage added by a third pattern is approximately that which would be added by placing the center of the third pattern at the previously computed distance, x, from the center of the circle which replaces the first two patterns. This configuration is in turn replaced by a new circle whose area equals the coverage expected from the first three patterns, etc.

While these assumptions leave much to be desired, the one experimentally verified case is in very good agreement.

Bases of the Approximations

Suppose the center of a circular pattern of radius a falls at a distance x (see Figure 1) from the center of a previously placed circle of radius r_{n-1}. Designating the additional area covered by

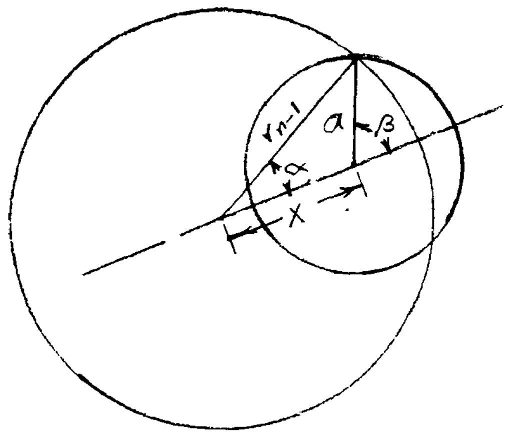

Figure 1

this new pattern by C_n ,

$$C_n = 0 \quad \text{if} \quad a + x \leq r_{n-1}$$

$$= \pi a^2 \quad \text{if} \quad x - a \geq r_{n-1}$$

$$= a^2 \beta_n - r_{n-1}^2 \alpha_n + a\, r_{n-1} \sin(\beta_n - \alpha_n) \qquad (1)$$

$$\text{if } r_{n-1} - a \leq x \leq r_{n-1} + a$$

where

$$\cos \alpha_n = \frac{r_{n-1}^2 + x^2 - a^2}{2\, x\, r_{n-1}} \qquad (2)$$

and

$$\cos \beta_n = \frac{r_{n-1}^2 - x^2 - a^2}{2\, a\, x} \quad . \qquad (3)$$

After each new circular pattern (of radius a) is placed, let the configuration be replaced by a single circle of the same total area. Designating the radius of this circle by r_n ,

$$\pi r_n^2 = \pi r_{n-1}^2 + C_n \quad . \qquad (4)$$

Equations 1, 2, and 3 are then applied, with the subscripts advanced by 1, to obtain C_{n+1}, etc.

Determination of x

If the radius of the first pattern placed is $r_1 = a$, then

$$C_1 = \pi a^2 . \tag{5}$$

As shown in D-447, the additional coverage to be expected from a second circular pattern aimed at the same point as was the first, is

$$E(C_2) = \pi a^2 e^{-a^2/2s^2} \left[I_0 \left(\frac{a^2}{2s^2} \right) + I_1 \left(\frac{a^2}{2s^2} \right) \right] \tag{6}$$

The separation, x, which would give the same coverage is determined by equating this coverage to

$$C_2 = a^2 [\pi - 2\alpha_2 + \sin 2\alpha_2], \tag{7}$$

where

$$x = 2a \cos \alpha_2 . \tag{8}$$

Table I shows the numerical relationship between a/s, $C_2/\pi a^2$, and x/s for a limited range of values. It may be observed that, while the root-mean-square distance between the centers of patterns individually aimed at the same point with the same dispersion is twice the standard deviation, x/s is always less than two.

Limitation

It can be seen that the estimate of the total coverage provided by this approximation cannot exceed $\pi(a + x)^2$, no matter how large n may be.

TABLE I

a/s	$C_2/\pi a^2$	x/s
1.0	0.80	1.36
1.2	0.74	1.50
1.4	0.68	1.58
1.6	0.62	1.63
1.8	0.57	1.67
2.0	0.52	1.70
2.2	0.48	1.71

Tables

Table II gives the relative effectiveness (that is, the ratio of C_n to πa^2) for a number of cases in which x remains constant throughout.

Table III gives the cumulative value of the effectiveness of n circles. This equals the ratio of the total area covered by n circles, to that covered by a single circle.

TABLE II

The Relative Coverage ($C_n/\pi a^2$) Obtained from n^{th} Circle

	x/a			
n	0.5	0.6	0.7	0.8
1	1.0000	1.0000	1.0000	1.0000
2	.3150	.3762	.4364	.4954
3	.1963	.2340	.2888	.3294
4	.1351	.1689	.1998	.2362
5	.0985	.1243	.1488	.1775
6	.0748	.0950	.1147	.1378
7	.0584	.0746	.0908	.1096
8	.0467	.0600	.0734	.0890
9	.0380	.0490	.0603	.0734
10	.0315	.0408	.0503	.0614

TABLE III

The Ratio of the Total Coverage Obtained with n Circles to the Area Covered by a Single Circle

	x/a			
n	0.5	0.6	0.7	0.8
1	1.0000	1.0000	1.0000	1.0000
2	1.3150	1.3762	1.4364	1.4954
3	1.5112	1.6102	1.7252	1.8248
4	1.6463	1.7791	1.9250	2.0610
5	1.7449	1.9034	2.0737	2.2385
6	1.8196	1.9984	2.1884	2.3762
7	1.8780	2.0740	2.2793	2.4859
8	1.9247	2.1340	2.3527	2.5749
9	1.9628	2.1830	2.4130	2.6483
10	1.9942	2.2238	2.4633	2.7098

Experimental Verification

No data are at hand as to the effectiveness of coverage when circular patterns are employed. The nearest approach to experimental data is furnished by the RAND Coverage machine, which employs square patterns. An example has been selected in which the mean radial error is equal to one-fourth the length of one side of the pattern. In this case the probability of a pattern falling partly outside the target area is so small that it may be neglected for the purposes of this comparison.

Exact theory (see RM-133) shows that the expected overlap when two squares of the same size are independently aimed at the same point with the same variance, s^2, is

$$E(\text{Overlap}) = 4\left\{2a\ A(2a/\sigma) - \sigma\left[\phi(0) - \phi(2a/\sigma)\right]\right\}^2$$

where $2a$ = length of the side of the square

$$\sigma^2 = 2s^2$$

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$A(x) = \int_0^x \phi(t)dt\ .$$

Since the mean radial error is

$$\bar{r} = \frac{a}{2}\ ,$$

$$\sigma^2 = 2s^2 = \frac{4}{\pi}\bar{r}^2 = a^2/\pi$$

and

$$E(\text{Overlap}) = 2.4022a^2.$$

The expected additional coverage from the second square is $4a^2 - 2.4022a^2$ or $1.5978a^2$. The effectiveness of this second square is therefore $1.5978a^2/4a^2$ or 0.3994. The value obtained experimentally in this case was 0.3975.

Table IV compares the cumulative effectiveness of the square pattern coverage with the values obtained by linear interpolation in Table III, the interpolation being such as to make $C_2 = 0.3995$. This comparison is shown graphically in Figure 2.

Considering the crude nature of the approximations involved, and that the experimental verification employs square patterns instead of the circular ones on which the approximations are based, the agreement is surprisingly high. It is wondered whether further comparison with experimental results will continue to show as good agreement.

TABLE IV

Cumulative Effectiveness, or the Ratio of the Total Area Covered by n Patterns to that Covered by a Single Pattern

n	By experiment with square patterns	By interpolation for circular patterns
1	1.000	1.000
2	1.398	1.399
3	1.642	1.655
4	1.809	1.836
5	1.938	1.969
6	2.040	2.072
7	2.132	2.153
8	2.209	2.218
9	2.278	2.272
10	2.340	2.316

HHG:rb