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ON THE OPTIMAL USE OF GUIDED MISSILES—I: ALLOCATION OF MISSILES

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SUMMARY

This is the first of a series of papers devoted to mathematical problems arising in the use of guided missiles. The paper deals with the optimal allocation of missiles against a given target system. It is shown that the computational solution of these problems can be greatly simplified by means of the functional-equation technique of dynamic programming.

CONTENTS

ON THE OPTIMAL USE OF GUIDED MISSILES—I: ALLOCATION OF MISSILES

1. INTRODUCTION

This is the first of a series of papers devoted to various mathematical problems that arise in the use of guided missiles against enemy targets, which may be stationary or attacking missiles or planes. In this first note we shall consider the general problem of the allocation of missiles to the bombardment of fixed enemy targets.

This is a problem that we discussed some years ago [2]. Since then we have found that the theory of dynamic programming [3] may be utilized to reduce the computational problems involved to a very simple level. Furthermore, the computational scheme we propose automatically yields a "sensitivity analysis" of the solution (cf. [1] for further discussion).

The method we propose is applicable to a large class of problems involving the allocation of resources, both economic and military, as we shall show in subsequent papers.

The mathematical problems arising from questions concerning the strategic and tactical use of guided missiles are simplified by the assumption that the missiles are expendable; there are no "empties," or decoys, in the model we are considering.

2. A SIMPLE MATHEMATICAL MODEL

The first model we shall consider is the following. There are N targets, of values $v_1, v_2, \ldots, v_N$, respectively, with independent defenses against guided missiles. We have a supply of S identical missiles, which we wish to use so as to maximize the damage we achieve with these missiles. Since this damage is a stochastic quantity, we must agree to use some average damage as the measure of the value of a policy. The two most common and useful criteria are the following:

a. Total expected damage.

b. Probability that the damage is no less than some preassigned amount D.

We shall consider both of these.

3. EXPECTED DAMAGE

Let us consider first the problem of maximizing the expected damage. A number S_i of the S missiles are allocated to the destruction of the i^{th} target. If $P_i(S_i)$ is the probability that the i^{th} target will be destroyed by these S_i missiles, then the total expected damage is

$$(1) \qquad E_N(D) = \sum_{i=1}^{N} P_i(S_i) v_i \ .$$

We wish to maximize this quantity subject to the restrictions

$$(2) \qquad \sum_{i=1}^{N} S_i = S \ , \qquad S_i = 0, 1, 2, \ldots, S.$$

Observe that we have taken the probability of destruction of the i^{th} target to have the form $P_i(S_i)$. By this we wish to indicate that the probability depends not only on the number of missiles allocated, but also upon the target, both by virtue of its defenses, and also because of the fact that different launching sites may be used for different targets.

As the problem stands, a direct application of classical variational techniques is blocked by at least two facts. The first is that the S_i assume only integral values; the second is that the $P_i(S_i)$ may not be well enough known to be differentiated. Bypassing these two obstacles, we note that the use of calculus is still made difficult by the fact that some of the S_i may be zero—i.e., on the boundary of the region of variation.

Let us then present another method, an application of general techniques in the theory of dynamic programming, which reduces the determination of the maximum in (1) to a sequence of N one-dimensional maximizations. The computational solution may then be obtained in a very simple fashion either by hand computation or on a digital computer.*

4. THE FUNCTIONAL-EQUATION APPROACH

Observe that the maximum value of $E_N(D)$ depends only upon the initial number S of missiles and the particular set of N targets, since we have taken the other parameters and functions

*The details of this computation are given in a later paper of the series, RM-1747.

as fixed.

Hence we define

(1) $$f_N(S) = \max_{\{S_i\}} E_N(D).$$

Assume that we have chosen S_N, the allocation of missiles to the N^{th} target. The problem that remains is that of utilizing the remaining $S - S_N$ missiles so as to achieve the maximum expected damage against the remaining $(N - 1)$ targets. Hence, for any choice of S_N, the expected damage is

(2) $$p_N(S_N)v_N + f_{N-1}(S - S_N).$$

It follows that S_N must be chosen to maximize the expression (2). Hence the recurrence relation is

(3) $$f_N(S) = \max_{0 \leq S_N \leq S} [\, p_N(S_N)v_N + f_{N-1}(S-S_N) \,]$$

for $N = 2, 3, \ldots$, with

(4) $$f_1(S) = p_1(S)v_1 .$$

The variable S_N assumes only zero or integer values 0, 1, 2, ..., S.

We have thus reduced the N-dimensional problem to a sequence of N one-dimensional problems of exceedingly simple type. In the course of computing the solution, we automatically compute the sequence $\{f_k(S)\}$, $k = 1, 2, \ldots, N$, thus obtaining information that is useful in itself: the process yields sensitivity information relative both to S and to N.

5. A PARTICULAR CASE

A case of particular interest is that where the probability that the target is destroyed by an individual missile depends upon the number fired at the target at one time, and where we assume that the survival of any individual missile is independent of the fate of the others.

Let us set

(1) $P_i(k)$ = the probability that any one of k missiles fired simultaneously at the i^{th} target destroys the target.

Then the probability that the target is destroyed by a salvo of k missiles, neglecting partial or cumulative damage, is

(2) $$p(k) = 1 - (1 - P_i(k))^k.$$

An important problem that arises before we can compute $P_i(S_i)$ is that of determining our salvo policy. Shall we fire these S_i missiles all at the same time, one at a time, or in some compromise fashion? This we shall discuss in the next section.

6. SALVO POLICY

Let $p(k)$ be the probability of destroying the target when k missiles are fired simultaneously. Let us demonstrate:

If $(1 - p(k))^{1/k}$ is monotone decreasing, the optimal procedure consists of firing all the missiles at once. If $(1 - p(k))^{1/k}$ is monotone increasing, the optimal procedure consists of firing the missiles one at a time.

Proof. Let us proceed inductively. Let $f(n)$ denote the probability of destroying the target with n missiles when an optimal salvo policy is used. Then

$$(1)\qquad f(n+1) = \max_{1\leq k\leq n+1} \left[p(k) + (1-p(k))f(n+1-k) \right].$$

Assume to begin with that $(1-p(k))^{1/k}$ is monotone decreasing and the result holds for $k = 1, 2, \ldots, n$. Then (1) yields

$$(2)\qquad f(n+1) = \max_{1\leq k\leq n+1} \left[p(k) + (1-p(k))p(n+1-k) \right].$$

Set $q(k) = 1 - p(k)$. Then

$$(3)\qquad f(n+1) = 1 - \min_{1\leq k\leq n+1} q(k)q(n+1-k).$$

With $q(k) = b(k)^k$, this becomes

$$(4)\qquad f(n+1) = 1 - \min_{1\leq k\leq n+1} b(k)^k\, b(n+1-k)^{n+1-k}.$$

If $b(k)$ is monotone decreasing, we have

$$(5)\qquad b(k)^k\, b(n+1-k)^{n+1-k} \geq b(n+1)^{n+1}.$$

This minimum value is attained only for $k = n + 1$.

The same type of proof holds in the other case where $(1-p(k))^{1/k}$ is monotone increasing in k.

7. EXAMPLE

In many models it is assumed that the probability of an individual missile destroying the target is $e^{-c_1/k}$, where c_1 depends upon the target. In this case

(1) $$p(k) = 1 - (1 - e^{-c_1/k})^k,$$

and thus

(2) $$(1 - p(k))^{1/k} = 1 - e^{-c_1/k},$$

which is monotone decreasing as k increases. Hence the optimal policy is to fire all missiles at once.

8. PROBABILITY OF OBTAINING DAMAGE

Let us now consider the corresponding equations we obtain if we take as a criterion the probability of achieving a damage D. Let

(1) $f_N(D,S)$ = the probability that a damage greater than or equal to D is attained against the N targets given S missiles and using an optimal policy.

We have

(2) $$f_1(D,S) = p_1(S), \quad v_1 \geq D$$
$$= 0, \quad 0 \leq v_1 < D,$$

and

(3) $$f_N(D,S) = \max_{0 \leq S_N \leq S} \left[p_N(S_N) f_{N-1}(D - v_N, S - S_N) + (1 - p_N(S_N)) f_{N-1}(D, S - S_N) \right]$$

for N = 2, 3, ...

This is still a relatively simple computation*.

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*Details of the computation may be found in a later paper in the series, RM-1748.

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