U. S. AIR FORCE

PROJECT RAND

RESEARCH MEMORANDUM

OPTIMAL EMPLOYMENT OF TACTICAL AIR FORCES IN THEATER AIR TASKS—II: A GAME-THEORETIC ANALYSIS

by

Leonard D. Berkovitz
Melvin Dresher

RM-2137

ASTIA Document Number AD 150684

March 28, 1958

Assigned to ______________________

The RAND Corporation
1700 MAIN ST. • SANTA MONICA • CALIFORNIA

SUMMARY

The general problem of allocating aircraft among three different air tasks—counter air, air defense, and support of ground operations—in a multistrike campaign is analyzed as a two-sided war game. As in previous papers, it is assumed that counter-air missions destroy enemy parked aircraft, air-defense missions reduce enemy counter-air operations, and support of ground operations determines the payoff of the campaign. In this paper, we show how the magnitude of the air-defense potential affects the optimal tactics.

The optimal allocations for this type of tactical air game can be summarized in terms of three time periods:

1. Terminal Stage. During the last strikes of the campaign, both sides—regardless of their strengths—concentrate their forces on support of ground operations.
2. Middle Stage. In each of these strikes prior to the terminal stage, the optimal tactics of a combatant depend on his force size relative to his opponent's. If he has a smaller force than his opponent, then he concentrates on counter-air missions. If he has a larger force than his opponent, then he splits his force between counter-air and ground-support operations. The duration of this middle stage increases with decreasing air-defense potential.
3. Early Stage. During the early strikes of the campaign, the stronger combatant splits his forces either between

two tasks, counter air and air defense, or among all three tasks. At the same time, the weaker side chooses a task at random, subject to a given probability distribution, and concentrates his forces on that task.

If we examine the use of air defense in the optimal tactics, we find that air-defense allocations are relatively infrequent. The air-defense task is neglected during the middle and terminal stages of the campaign. Allocations to air defense are made only in the early strikes of a long campaign. Further, the weaker combatant uses the air-defense task only as a bluffing tactic and with small probability. If the campaign is short or if the defense potential is small, then the air-defense task is totally neglected by both sides.

CONTENTS

OPTIMAL EMPLOYMENT OF TACTICAL AIR FORCES IN THEATER AIR TASKS—II: A GAME-THEORETIC ANALYSIS

1. INTRODUCTION

The problem of allocating tactical aircraft among three air tasks—counter air, air defense, and support of ground operations—has been studied analytically as a two-sided war game. In [1], we formulated this allocation problem as a two-person zero-sum game and summarized the optimal strategies for certain special values of the parameters. In [2], we presented mathematical derivations of these optimal strategies.

In both of the above papers it was assumed that each plane allocated to air defense prevents one attacking plane from fulfilling its counter-air mission. Although unrealistic, this was a convenient assumption in order to obtain insight into the general form of the optimal strategies in a three-task war game.

In the present paper, we assume an arbitrary value for the air-defense potential and derive the set of optimal strategies as a function of the air-defense potential.

2. DESCRIPTION OF TACTICAL GAME

The tactical air campaign is assumed to consist of N strikes, or moves, each strike being an allocation of forces among counter air, air defense, and close support. It is convenient to count strikes from the end of the campaign. Thus, when we speak of the i-th strike of a campaign, we mean that we are at the i-th strike from the end—or, put another way, that i strikes are left in the campaign. Let p_i and q_i be the force sizes of Blue and Red, respectively, at the start of the i-th strike. Each side makes its allocation knowing the force strength of the enemy, but without knowing the opponent's allocation. We assume that all planes can fly and that there are no aborts. Suppose that, during the i-th strike of the campaign, Blue allocates x_i planes on counter-air missions, u_i planes on air-defense missions, and the remainder, $p_i - x_i - u_i$, on close-support missions. Suppose that, simultaneously, Red allocates y_i planes on counter-air missions, w_i planes on air-defense missions, and $q_i - y_i - w_i$ on close-support missions.

If we further assume that it requires $c \geq 1$ air-defense planes to prevent one attacking plane from reaching target, and let $k = 1/c$, where $k \leq 1$ is the air-defense potential, then on the i-th strike the number of Blue planes that penetrate Red's defenses is

$$\max (0, x_i - kw_i).$$

The number of Red planes that penetrate Blue's defenses is

$$\max (0, y_i - ku_i).$$

Let us assume that each attacking plane that penetrates the opponent's defenses can destroy one plane on the ground. Further, let us assume that Blue planes used in air defense survive and that Red aircraft that fail to penetrate Blue defenses return to base. Then the number of Blue planes surviving the i-th strike, or the number of planes available for the next strike, is

$$p_{i-1} = \max\left[0,\ p_i - \max(0,\ y_i - ku_i)\right]. \tag{1}$$

Similarly, Red's force for the next strike will be given by

$$q_{i-1} = \max\left[0,\ q_i - \max(0,\ x_i - kw_i)\right]. \tag{2}$$

These forces, p_{i-1} and q_{i-1}, are allocated among the three air tasks by Red and Blue on the (i-1)-st strike. The process is repeated until N strikes have been made by each side.

As in the previous papers, we assume the payoff to Blue for the campaign, or game, is the N-strike total of Blue's superiority over Red in close-support effort, or

$$\text{Payoff} = \sum_{i=1}^{N}\left[(p_i - x_i - u_i) - (q_i - y_i - w_i)\right]. \tag{3}$$

In what follows, we shall give a description of the optimal strategies for each of the players. We shall see that in general the campaign can be divided into three time periods, such that in each of these time periods the optimal tactics are similar. These time periods will be referred to as ground-support phase, counter-air phase, and randomization phase.

Before proceeding to describe the solution of the game, we make the following comment. Suppose that at the start of a campaign of i strikes, Blue has p_i planes and Red has q_i planes. Let $v_i(p_i, q_i)$ denote the value of the game defined by this campaign. We showed in [2] that in order to obtain optimal strategies in a campaign of N strikes, it is sufficient to solve the game of which the payoff is given by

$$M_i(x_i,u_i;y_i,w_i) = (p_i - x_i - u_i) - (q_i - y_i - w_i) + v_{i-1}(p_{i-1}, q_{i-1}) \tag{4}$$

for each i, $1 \leq i \leq N$, where p_{i-1} and q_{i-1} are defined by (1) and (2), respectively.

3. GROUND-SUPPORT PHASE

First, we shall show that optimal allocations at the end of the campaign require a series of strikes on ground support; i.e., during the last strikes of the campaign Blue and Red dispatch all their planes on close-support missions. In our case, in which we assume that all planes can fly and there are no aborts, both sides concentrate on close support during the last two strikes of the campaign.

It is readily apparent that

$$\max_{x_1,u_1} \min_{y_1,w_1} M_1(x_1,u_1;y_1,w_1) = \min_{y_1,w_1} \max_{x_1,u_1} M_1(x_1,u_1;y_1,w_1) = p_1 - q_1.$$

Hence we have

$$v_1(p_1, q_1) = p_1 - q_1,$$

and the optimal allocation at the last $(i=1)$ strike is

$$x_1^* = u_1^* = y_1^* = w_1^* = 0.$$

At the next to the last strike (i=2), we have

(5) $$M_2(x_2,u_2;y_2,w_2) = p_2 - q_2 - x_2 - u_2 + y_2 + w_2 + \max\left[0,\ p_2 - \max(0,\ y_2 - ku_2)\right] - \max\left[0,\ q_2 - \max(0,\ x_2 - kw_2)\right].$$

Substituting particular strategies in (5), we can verify that

$$M_2(0,\ 0;\ y_2,\ w_2) \geq 2(p_2 - q_2) \quad \text{for all} \quad y_2,\ w_2,$$

and that

$$M_2(x_2,\ u_2;\ 0,\ 0) \leq 2(p_2 - q_2) \quad \text{for all} \quad x_2,\ u_2.$$

Hence the optimal allocation at $i = 2$ is again

$$x_2^* = u_2^* = y_2^* = w_2^* = 0,$$

and we obtain

$$v_2(p_2,\ q_2) = 2(p_2 - q_2).$$

We thus have the following result:

Optimal Tactics During Terminal Period—During the last two strikes of a campaign, Red and Blue allocate all their resources to close support.

4. COUNTER-AIR PHASE

We shall show that preceding the ground-support phase there is a counter-air phase; i.e., both sides make counter-air allocations during this period. Specifically, the weaker side allocates all his resources to counter-air missions, while the stronger side allocates part of his forces to counter-air missions and the remainder to close-support missions.

In order to determine the optimal tactics for $i = 3$, it is necessary to express the payoff $M_3(x, u; y, w)$ in the following six pieces, where we drop the subscript 3 from the strategic variables:

$$(6)\quad \begin{array}{lll} 3p_3-3q_3-x-u+y+w & \text{if } y-ku \le 0 & \text{and } x-kw \le 0, \\ 3p_3-q_3-x-u+y+w & \text{if } y-ku \le 0 & \text{and } x-kw \ge q_3, \\ 3p_3-3q_3+x-u+y+(1-2k)w & \text{if } y-ku \le 0 & \text{and } 0 \le x-kw \le q_3, \\ 3p_3-3q_3-x-(1-2k)u-y+w & \text{if } y-ku \ge 0 & \text{and } x-kw \le 0, \\ 3p_3-q_3-x-(1-2k)u-y+w & \text{if } y-ku \ge 0 & \text{and } x-kw \ge q_3, \\ 3p_3-3q_3+x-(1-2k)u-y+(1-2k)w & \text{if } y-ku \ge 0 & \text{and } 0 \le x-kw \le q_3. \end{array}$$

Let us assume that Blue is stronger at strike $i = 3$—that is, that $p_3 \ge q_3$. Then, if $k \le 1/2$, it can be verified that for (6) we have

$$\max_{x,u} \min_{y,w} M_3(x, u; y, w) = \min_{y,w} \max_{x,u} M_3(x, u; y, w)$$

$$= 3(p_3 - q_3).$$

Hence, for $i = 3$ the optimal tactics are pure, and will be given by

$$x_3^* = q_3, \quad u_3^* = 0,$$
$$y_3^* = q_3, \quad w_3^* = 0.$$

The preceding tactics are also optimal for earlier strikes. Let τ be the largest integer contained in $1/k$, or $\tau = [1/k]$. Then we can show by induction that for all i satisfying $2 < i \leq \tau + 1$ we have

$$\max_{x,u} \min_{y,w} M_i = \min_{y,w} \max_{x,u} M_i = i(p_i - q_i)$$

and

$$x_i^* = q_i, \quad u_i^* = 0,$$
$$y_i^* = q_i, \quad w_i^* = 0.$$

This applies for $\tau - 1$ strikes. In particular, if $k > 1/2$ then the counter-air phase is vacuous.

Accordingly, for the counter-air phase, the result is as follows:

<u>Optimal Tactics During Middle Period</u>—During each of the $\tau - 1$ strikes, $i = \tau + 1, \tau, \tau - 1, \ldots, 4, 3,$ the weaker side allocates all his resources, q_i, to the counter-air missions. At the same time, the stronger side allocates q_i of his resources to counter air and the remainder $p_i - q_i$ to close support.

5. RANDOMIZATION PHASE

In the previous section, we have shown that during the last $\tau + 1$ strikes the optimal tactics for both sides are pure—neither side bluffs (i.e., neither side uses a mixed strategy). However, if $i \geq \tau + 2$, then, depending on the relative strengths of the two sides, it may be necessary for the weaker side to bluff; we shall show, however, that the stronger side never needs to bluff.

Let us consider the strike $i = \tau + 2$. Then we have the following payoff (where the subscript $\tau + 2$ is omitted):

$$(7) \qquad M(x,u;y,w) = (p - x - u) - (q - y - w) + (\tau + 1) \max \left[0,\ p - \max(0,\ y - ku)\right] - (\tau + 1) \max \left[0,\ q - \max(0,\ x - kw)\right].$$

The solution to (7) breaks into two cases according to the relative sizes of the forces p and q.

One can verify, by substituting in (7), that if $1 \leq p/q \leq 1/k$ then the value of the game is

$$v = (\tau - 1)(k + 1)(p - q),$$

and both sides have pure optimal tactics, as follows:

$$x^* = q, \quad u^* = p - q,$$
$$y^* = q, \quad u^* = 0.$$

On the other hand, if $p/q \geq 1/k$, then

$$v = (\tau + 2)p - \frac{(\tau + 1)k^2 + 1}{k} q.$$

Blue splits his forces among the three tasks while Red, the weaker side, uses a mixed strategy. The optimal allocation for Blue is

$$x^* = q, \quad u^* = q(\frac{1}{k} - 1).$$

The optimal choice for Red is

$$y^* = q, \quad w^* = 0 \quad \text{with probability} \quad \frac{1}{k(\tau + 1)},$$

$$y^* = 0, \quad w^* = q \quad \text{with probability} \quad 1 - \frac{1}{k(\tau + 1)}.$$

We observe that we have $x^* + u^* \leq p$.

At strike $i = \tau + 3$, there are three cases to consider, depending on the relative sizes of forces. Again, omitting the subscript $\tau + 3$, one can verify the following by substitution in the payoff function:

(a) If

$$1 < \frac{p}{q} \leq 1 + \left(\frac{\tau + 1}{\tau + 2}\right)\left(\frac{1 - k^2}{k}\right),$$

then

$$v = (\tau + 2)(1 + k)(p - q),$$

and both sides have pure optimal tactics given by

$$x^* = q, \quad u^* = p - q,$$
$$y^* = q, \quad w^* = 0.$$

(b) If

$$1 + \frac{\tau + 1}{\tau + 2}\,\frac{1 - k^2}{k} \leq \frac{p}{q} \leq 1 + k + \frac{\tau + 1}{k(\tau + 2)} - \frac{k}{(\tau + 1)k^2 + 1},$$

then

$$v = \frac{\tau + 2}{\mu + (\tau + 2)k}\left[(\mu + \tau k + 2k + \mu k)p - \mu(1 + k + k^2)q\right],$$

where

$$\mu = (\tau + 2)k - k + \frac{1}{k}.$$

Blue's optimal allocation is

$$x^* = \frac{(\tau + 2)kp + (\mu - \tau - 2 + \mu k)q}{\mu + \tau k + 2k},$$

$$u^* = \frac{\mu p - (\mu - \tau - 2 + \mu k)q}{\mu + \tau k + 2k}.$$

Red's optimal tactics are mixed. He allocates

$$y^* = q, \quad w^* = 0 \quad \text{with probability} \quad \frac{\mu}{\mu + \tau k + 2k},$$

$$y^* = 0, \quad w^* = q \quad \text{with probability} \quad \frac{\tau k + 2k}{\mu + \tau k + 2k}.$$

Observe that, as in case (a), we have $x^* + u^* = p$.

(c) If

$$\frac{p}{q} \geq 1 + k + \frac{\tau + 1}{k(\tau + 2)} - \frac{k}{(\tau + 1)k^2 + 1},$$

then

$$v = (\tau + 3)p - \left(2 + k + \frac{1}{k} - \frac{1}{\mu} - \frac{1}{\tau k + 2k}\right)q.$$

Blue's optimal tactic is

$$x^* = (1 + k - \frac{1}{\mu})q,$$
$$u^* = \frac{\tau + 1}{(\tau + 2)k}\, q.$$

Red's optimal tactics are mixed:

$$y^* = q, \quad w^* = 0 \quad \text{with probability} \quad \frac{1}{(\tau + 2)k},$$

$$y^* = 0, \quad w^* = q \quad \text{with probability} \quad \frac{1}{\mu},$$

$$y^* = 0, \quad w^* = 0 \quad \text{with probability} \quad 1 - \frac{1}{\mu} - \frac{1}{(\tau + 2)k}.$$

Observe that now we have $x^* + u^* \leq p$.

Using methods and arguments almost identical with those used in [2], we can determine the optimal tactics for all the remaining strikes of the campaign, namely those for which i satisfies $\tau + 4 \leq i \leq N$. For each such value of i, the optimal tactics break down into $i - \tau$ cases on the ratio p_i/q_i. For each i, we compute a unique sequence of $i - \tau - 1$ numbers, $r_1 < r_2 < r_3 < \ldots < r_{i-\tau-1}$, where each r is larger than 1 and its value depends only on k and i, and $r_{i-\tau-1}$

exceeds 2. The optimal tactics depend on the relative strengths of forces, in the following manner:

(a) If

$$1 \leq \frac{p_i}{q_i} \leq r_1,$$

then Blue's optimal tactics are

$$x_i^* = q_i, \quad u_i^* = p_i - q_i.$$

Red's optimal tactics are

$$y_i^* = q_i, \quad w_i^* = 0.$$

(b) If

$$r_j \leq \frac{p_i}{q_i} \leq r_{j+1}, \quad i \leq j \leq i - \tau - 1,$$

Blue has a unique optimal allocation (x_i^*, u_i^*) with the property that

$$x_i^* > q_i, \quad u_i^* = p_i - x_i^*.$$

The value of x_i^*, of course, varies with j. Red's optimal choice is

$$y_i^* = q_i, \quad w_i^* = 0 \text{ with probability } \beta_j > 0,$$

$$y_i^* = 0, \quad w_i^* = q_i \text{ with probability } 1 - \beta_j.$$

(c) If

$$\frac{p_i}{q_i} > r_{i-\tau-1},$$

then Blue has a unique optimal allocation (x_i^*, u_i^*), with the properties

$$x_i^* > q_i, \quad u_i^* > q_i, \quad x_i^* + u_i^* < p_i.$$

Red's optimal choice is

$$y_i^* = q_i, \quad w_i^* = 0 \quad \text{with probability} \quad \gamma_i > 0,$$

$$y_i^* = 0, \quad w_i^* = q_i \quad \text{with probability} \quad \delta_i > 0,$$

$$y_i^* = 0, \quad w_i^* = 0 \quad \text{with probability} \quad 1 - \gamma_i - \delta_i > 0.$$

We summarize these results as follows:

Optimal Tactics During Early Period—During each of the following $n - \tau - 1$ strikes, $n, n - 1, n - 2, \ldots, \tau + 3, \tau + 2$, the stronger side splits his forces either between counter air and air defense or among all three tasks. The latter alternative occurs when he is very strong. During these strikes, the weaker player does one of the following three things, depending on his relative weakness:

1. If he is slightly weaker than his opponent, then he concentrates his resources on counter air.
2. If he is moderately weaker than his

opponent, then he concentrates either on counter air or air defense, with the particular concentration chosen at random subject to a given probability distribution.

3. If he is very weak, he concentrates his resources on one of the three tasks chosen at random subject to a given probability distribution.

REFERENCES

1. Berkovitz, Leonard D., and Melvin Dresher, Optimal Employment of Tactical Air Forces in Theater Air Tasks—A Game-Theoretic Analysis, The RAND Corporation, Research Memorandum RM-1877, February 21, 1957.

2. Berkovitz, Leonard D., and Melvin Dresher, A Multimove Infinite Game with Linear Payoff, The RAND Corporation, Paper P-1151, February 21, 1958.