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RESEARCH MEMORANDUM

A GAME SOLUTION TO A MISSILE LAUNCHING SCHEDULING PROBLEM

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SUMMARY

A missile launching scheduling problem is described and solved as a game. Blue must launch his missiles within a certain time period after outbreak of war with Red. Each Blue site can launch one missile at a time. If a Red missile hits near the site, it destroys any Blue missile being launched at that time. Both sides know each other's initial missile strength.

Blue wants to maximize (and Red wants to minimize) the expected number of successful Blue missile launchings. This multistage continuous game fortunately has a discrete solution with both sides playing at random combinations of times chosen from finite sets of time. The solution is not affected by stagewise information received by either side concerning the current strength of the enemy forces.

A more complicated extension involves a payoff to Blue as a decreasing function of launching time.

CONTENTS

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1. INTRODUCTION

An atomic war* has just broken out between Blue and Red. Blue must launch all his missiles within the next T hours. Blue can launch his missiles one at a time from a given launching site. While in the process of being launched, the Blue missile is vulnerable to enemy attack and is destroyed if a Red missile hits near the site during the launching period. It is assumed that the site and the other missiles (stored underground) are not destroyed.

Both sides know the initial size of the opposing forces. Blue can make stagewise decisions based on the current Red and Blue strength, but Red learns nothing of Blue's launching schedule. This can be interpreted, for example, as expressing Red's prior commitment to a firing schedule. Further, it turns out that the optimal strategies in this model are independent of any stagewise information concerning enemy forces.

Red tries to time his missile strikes so as to maximize the expected number of Blue missiles destroyed. Blue must schedule the launching of his missiles, all of which must be launched during the given time period so as to minimize this payoff.

Even though the game appears to be a multistage continuous game, fortunately there is a discrete solution simple to describe. Moreover, Blue cannot utilize his stagewise information. The game solution involves each side choosing at

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*This problem was suggested by members of the Ballistic Missile Division of the Air Research and Development Command.

random a combination of times chosen from finite sets of times, the strategies being independent of the size of the opposing force.

We shall solve the problem first for the case of one Blue launching site and then generalize to any number of sites. In Section 3 there are further extensions. This paper thus generalizes a result of Melvin Dresher's, who solved a continuous game for one Blue and one Red missile.

2. CASE 1. ONE SITE ONLY

Notation: Let

T = the time before which Blue must launch his missiles.

L = the length of the launching period for each Blue missile.

n = [T/L], the greatest integer $\leq$ T/L.

b = the number of Blue missiles to be launched from the site, where $b \leq n$.

r = the number of Red missiles attacking the site, where $r \leq n$.

R = a Red strategy of choosing r times of arrival of his missiles at Blue's launching site $t_1 < t_2 < \cdots < t_r$ where $0 \leq t_1 \leq T$.

R_0 = the Red mixed strategy of choosing at random a combination of r times from the set $L-\epsilon$, $2(L-\epsilon), \cdots, n(L-\epsilon)$, where ϵ is an arbitrarily small positive number. Actually, for any positive ϵ for which $L-\epsilon > \frac{T}{n+1}$, the results of Theorem 1 hold, but the proof is slightly more complicated algebraically.

B = a Blue strategy of choosing b launching starts $s_1 < s_2 < \cdots < s_b$ where $s_i - s_{i-1} \geq L$ and $0 \leq s_i \leq T-L$, each start based on the Red and Blue schedules to date.

B_0 = the Blue mixed strategy of choosing at random a combination of b starting times from the set $(0, L, \cdots, (n-1)L)$.

If for some $i \leq r$, $j \leq b$, $0 \leq t_i - s_j < L$, the Blue missile corresponding to s_j is destroyed. Let

$M(R, B; T, L, r, b) = M(R, B)$ = the expected number of Blue missiles destroyed when Blue plays strategy B and Red plays R.

$V(T, L, r, b)$ = the game value.

<u>Theorem 1</u>.

(1) $$M(R, B_0) \leq M(R_0, B_0) = \frac{rb}{n} \leq M(R_0, B)$$

so that

(2) $$V(T, L, r, b) = \frac{rb}{n}.$$

Before proving Theorem 1, we shall discuss the nature of the optimal strategies for Red and Blue. Call the time interval $0 \leq t < L$ the first period, the time interval $L \leq t < 2L$

the second period, etc., with the last period $(n-1)L \leq t \leq T$. The total time interval is then divided into n time periods.

There will be C_r^n different sets of r periods chosen from n periods. Assign equal probabilities to each of these combinations. Red will choose at random one such combination. He will then assign his r missiles to these r periods and will launch them so as to arrive over Blue's site at the proper time, i.e., corresponding multiples of $L-\epsilon$. An equivalent way for Red to carry through this strategy is to sequentially choose each period with probability $p_{n'} = \frac{r'}{n'}$ where r' is the current number of Red missiles and where there are n' periods left.

Blue will pick at random a combination of b periods chosen from n, and will start the launching of a missile at the beginning of each of these b periods. To accomplish this choice, Blue sequentially picks the first period with probability $\frac{b}{n}$ and subsequent periods with probability $\frac{b'}{n'}$ where b' is the current number of Blue missiles left and n' is the number of periods left. This, of course, can be done in advance for both players.

<u>Proof of Theorem</u>: Let Blue play strategy B_0. Thus each possible Blue starting time 0, L, 2L, ..., (n-1)L has <u>equal</u> probability b/n of being chosen. Any Red strategy such that his arrival times $t_1, \ldots, t_r$ follow by less than L time units r <u>different</u> times in the set (0, L, ..., (n-1)L) will give a payoff of rb/n. This is clearly as much as Red can do against

B_0. Thus by the definition of R_0,

(3) $$M(R, B_0) \leq M(R_0, B_0) = \frac{rb}{n} .$$

Next, let Red play strategy R_0. We show that Blue playing against R_0 will not lose anything by choosing his b starting times from the set (0, L, ..., (n-1)L). For suppose Blue chooses s_1 in the first period such that $0 \leq s_1 \leq L - \epsilon$. This missile will be destroyed if a Red missile arrives at $L - \epsilon$. Moreover, any subsequent schedule for Blue that is feasible (i. e., enables Blue to launch his remaining missiles one at a time before time T) when $0 < s_1 < L - \epsilon$ is also feasible for $s_1 = 0$. Thus Blue cannot lose by playing $s_1 = 0$ if at all in this period.

Next, whether or not Blue chooses the first period and therefore $s_1 = 0$, his next decision comes at the beginning of the second period (right after Red's first possible arrival time has passed). Here $n' = n - 1$, $r' = r$ or $r - 1$, $b' = b$ or $b - 1$, as the case may be. Blue makes his decision whether to start a launching sometime in the second period based on the new set of values (n', r', b'). As before, Blue cannot lose by starting at the beginning of this period if at all in this period. By induction, he can limit his possible starting times to the set (0, L, 2L, ..., (n-1)L), the same set as for strategy B_0.

This set of possible starting times for Blue is matched one to one by Red possible arrival times when he plays R_0. We can consider these n pairs of times as n positions or points numbered from n down to 1 in terms of time periods from the end.

Define W(n, r, b) as the payoff of this finite game when Blue plays optimally against R_0. Let q_n be the probability that Blue plays n. The probability that Red plays n is $p_n = r/n$, a consequence of the definition of R_0. In fact, at every stage n', R_0 calls for $p_n = r'/n'$ where r' is the current number of Red missiles.

Then note the recurrence relation,

$$(4) \qquad \begin{aligned} W(n, r, b) = & \; q_n p_n[1 + W(n-1, r-1, b-1)] \\ & + q_n(1 - p_n)W(n - 1, r, b - 1) \\ & + (1 - q_n)p_n W(n - 1, r - 1, b) \\ & + (1 - q_n)(1 - p_n)W(n - 1, r, b). \end{aligned}$$

If $b = n$, $1 - q_n = 0$; if $r = n$, $1 - p_n = 0$, so that the pertinent functions W(n', r', b') are defined, i.e., $r' \leq n'$, $b' \leq n'$.

Next we prove by induction that

$$(5) \qquad W(n, r, b) = \frac{rb}{n} .$$

This is obvious for n = 1. Note also that W(1, 1, 0) = W(1, 0, 1) = 0, so that by (4) for n = 2 we have

$$W(2, 1, 1) = q_2 \tfrac{1}{2}(1 + 0) + (1 - q_2)\tfrac{1}{2}(0 + 1) = \tfrac{1}{2}$$

independent of q_2.

Now assume that

(6) $W(k, r, b) = \frac{rb}{k}$, $k = 1, 2, \ldots, n-1$, $b \le k$, $r \le k$.

It is clear that $W(n, n, b) = W(n, r, n) = \frac{br}{n}$, giving (5) here. For $r < n$, $b < n$, by (4) and (6) we have

(7) $$W(n, r, b) = \min_{0 \le q \le 1} \left\{ q\left(\frac{r}{n}\right)\left(1 + \frac{(r-1)(b-1)}{n-1}\right) + q\left(1 - \frac{r}{n}\right)\left(\frac{r(b-1)}{n-1}\right) + (1-q)\left(\frac{r}{n}\right)\left(\frac{(r-1)b}{n-1}\right) + (1-q)\left(1 - \frac{r}{n}\right)\left(\frac{rb}{n-1}\right) \right\}.$$

Write (7) as

(8) $$\begin{aligned} n(n-1)W(n, r, b) &= \min_q \left\{ q\left[r(n-1) + r(r-1)(b-1) + r(n-r)(b-1)\right] + (1-q)\left[r(r-1)b + rb(n-r)\right]\right\} \\ &= \min_q \left\{ r\left[q(n-1) + q(n-1)(b-1) + (1-q)b(n-1)\right]\right\} \\ &= rb(n-1) \end{aligned}$$

so that (5) holds independent of q. Thus

(9) $M(R_0, B_0) \le M(R_0, B)$,

and this combined with (3) gives Theorem 1.

Corollary 1. Optimal strategies R_0 and B_0 are independent of the size of the opposing force.

Corollary 2. Even if Red had the same sequential information that Blue has, R_0 and B_0 are still optimal strategies for this new game whose value is also $\frac{rb}{n}$.

To show this, let $\overline{W}(n, r, b)$ be the value for this new game. The new game matrix for (2, 1, 1)

	p	$1-p$
q	1	0
$1-q$	0	1

has the solution $p = 1/2$, $q = 1/2$. Thus $\overline{W}(2, 1, 1) = \frac{rb}{n} = \frac{1}{2} = \underline{W}(2, 1, 1)$ here. Following through the induction arguments, we arrive at the general game matrix and its solution,

	$p_n = \frac{r}{n}$	$1 - p_n = \frac{n-r}{n}$
$q_n = \frac{b}{n}$	$1 + \frac{(r-1)(b-1)}{n-1}$	$\frac{r(b-1)}{n-1}$
$1-q_n = \frac{n-b}{n}$	$\frac{(r-1)b}{n-1}$	$\frac{rb}{n-1}$

so that $\overline{W}(n, r, b) = \underline{W}(n, r, b) = \frac{rb}{n}$.

Also note that strategy B_0 is equivalent to choosing each point sequentially with probability equal to the ratio of the current number of missiles to be launched divided by the number of time periods from the end. Similarly for Red, as has been remarked earlier.

CASE 2. ANY NUMBER OF BLUE LAUNCHING SITES

Blue is to allocate his missiles to his sites and then be prepared to schedule launching in an optimal manner in case of war. Red and Blue know the total sizes of opposing forces.

Suppose Blue has allocated his missiles to his k sites in some fashion $(b_1, b_2, \ldots, b_k)$ where $\sum_{i=1}^{k} b_i = b$. Let Red allocate his force into $(r_1, r_2, \ldots, r_k)$ where $\sum_{i=1}^{k} r_i = r$, where r_i are sent to Blue's i-th launching site. Then there is a collection of one-site games to be played. For site i, by Corollary 1 each player can play at random combinations of b_i or r_i points out of the set of n points independent of the opposing player's strength. Thus the payoff of the multisite game is

$$\sum_{i=1}^{k} V_i(T, L, r_i, b_i) = \sum_{i=1}^{k} \frac{r_i b_i}{n} .$$

Thus if Blue allocates his missiles uniformly to his sites such that $b_i = b/k$, then the payoff is

$$\frac{b}{nk} \sum_{i=1}^{k} r_i = \frac{rb}{nk}$$

independent of Red's allocation. Similarly, if Red chooses $r_i = r/k$, he can assure the same payoff independent of Blue's allocation. Hence $\frac{rb}{nk}$ is the value of the game and the uniform allocations are optimal. If k does not divide b or r exactly, then the obvious strategies are to allocate as uniformly as possible.

3. POSSIBLE EXTENSIONS

1) Suppose that a Red missile arriving near a site while a Blue missile is being launched has a probability $\alpha < 1$ of destroying it. Then if $r \leq n$, $b \leq n$, the game solution is unchanged from Case 1. If $r > n$, the problem now has meaning. Here if $r = r'n + r''$, $0 \leq r'' < n$, Red would first assign r' missiles to each of the set of n arrival times and assign the

remaining r'' by choosing at random a combination of r'' times from the set of n possible times.

2) An interesting case is where Blue is given credit, say $f(s)$ where $f'(s) \leq 0$, for a successful missile launching whose starting time is s. The payoff is the sum of these values which Blue wants to maximize. The other rules from Case 1 are unchanged. Here each side plays over the same discrete sets of points as before.

To show this, consider that against Red playing some probability distribution over the n times given for R_0, Blue plays at the beginning of each period by the arguments given for Case 1 plus the fact that $f'(s) \leq 0$. Thus there is a dominating discrete game over n points as before, but now the strategies are more complicated.

If $b = 1$, $r \leq n$, the game solution can be worked out in terms of $f(s)$ either by directly solving an n-by-n game or recursively solving a series of 2-by-2 games. To illustrate, consider the case $n = 4$, $r = 1$, $b = 1$. If Red hits before Blue has started, Blue would start his launching immediately thereafter. Thus Blue can take advantage of his sequential information here. The game matrix in this example is

		Red			
		p_4	p_3	p_2	p_1
Blue	q_4	0	f_4	f_4	f_4
	q_3	f_3	0	f_3	f_3
	q_2	f_3	f_2	0	f_2
	q_1	f_3	f_2	f_1	0

where q_k is the probability of Blue playing k or the number after the one chosen by Red, whichever comes earlier.

This game could be solved directly, but one can also express its matrix as

	p_4	$1-p_4$
q_4	0	f_4
$1-q_4$	f_3	V(3,1,1)

.

Since in case Red does not choose point 4, he should play the remaining three points as if Blue did not choose point 4. This follows from the fact that if Blue did choose 4, the payoff is f_4 independent of Red's play over the last three points.

Thus given V(3, 1, 1), one can find V(4, 1, 1) by solving a 2-by-2 game. It is clear how one could solve V(n, 1, 1) recursively in this way.

This same method carries over to the case V(n, r, 1). We have the game matrix

	p_n	$1-p_n$
q_n	0	f_n
$1-q_n$	V(n-1,r-1,1)	V(n-1,r,1)

where q_n is the probability that Blue plays n.

This can be seen as follows: Red has no sequential information of what Blue has done so must play the subgames as if Blue still has not chosen his number.

This strategy will give the same payoff as any other Red strategy if Blue has already played an earlier number and will be optimal in case Blue has not yet played his number.

If $b > 1$, then this method of solving a series of 2-by-2 games will not apply, since Red does not know the value of b at a given stage. The direct solution of a very large game matrix of this kind gets quite complicated.

Recently the author has worked with Melvin Dresher on some more sophisticated models with reliability factors on Red missiles, underground kill probabilities, and pin down probabilities due to fallout effects. These models and their game solutions will be discussed in a future paper.