

MEMORANDUM
RM-3095-PR
APRIL 1962

CUMULATIVE PROBABILITY OF
DETECTION FOR TARGETS APPROACHING
A UNIFORMLY SCANNING SEARCH RADAR

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PREPARED FOR:
UNITED STATES AIR FORCE PROJECT RAND

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PREFACE

This report presents the results of one study carried out as part of RAND's continuing work on the theory and advanced technology of radar systems. The present study deals with the capability of search radars for detecting approaching targets. The work is applicable to the current problem of the detection of ballistic missiles approaching a defense radar. However, the results are in general form and will be useful in designing and assessing the performance of search radars in detecting aircraft as well as ballistic missiles.

SUMMARY

This Memorandum deals with the cumulative detection probability of a search radar when it is scanning uniformly. This is the probability that a target approaching the radar at a constant radial velocity is detected at least once by the time it reaches a given range, as distinguished from the more common blip-scan ratio (a single-scan detection probability). It is shown that for constant-velocity targets the range for a given cumulative detection probability varies as the cube root of the power-aperture product, rather than as the fourth root. Curves of cumulative detection probability as a function of normalized range are given for three different target scintillation models. Also, curves of optimum (normalized) frame time are given as a function of the desired cumulative detection probability for each of the three target scintillation characteristics.

ACKNOWLEDGMENTS

The authors are indebted to Frank Katayama for the calculations and curves and to H. H. Bailey, F. Tatum and J. Hult for helpful criticism.

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LIST OF SYMBOLS

A	effective antenna area
B	IF bandwidth
f_r	pulse repetition frequency
G	antenna gain
k	Boltzmann's constant
L	loss factor
N	number of pulses (samples) integrated incoherently
N_c	number of pulses integrated coherently
n	average number of noise samples between false alarms
\bar{P}	average transmitted power
P_c	cumulative detection probability
$p(R/R_o)$	blip-scan ratio
\hat{P}	peak transmitted power
Q_1	correction factor for R_1 --non-fluctuating target or scintillation model of Fig. 3
Q_2	correction factor for R_1 --target scintillation model of Fig. 4
R	range
R_1	a range defined in Eq. (4)
R_o	range for unity signal-to-noise ratio
T_d	dwelt time, or time a target is within the radar beam during a single scan
T_{eff}	effective receiver temperature
T_f	frame time
V_c	target closing velocity
v	R/R_1

Δ	radial distance traversed by target during a search frame
δ	Δ/R_1
σ	target echo area
Ω	solid angle of search frame
ω	solid angle of radar beam

I. INTRODUCTION

In describing the capability of a search radar, one is often interested in the cumulative detection probability, i.e., the probability that an approaching target will have been detected at least once by the time it reaches a given range. This is to be distinguished from the probability of detecting a target at a given range during a single scan. The latter, which is sometimes called the blip-scan ratio, has been calculated and presented in convenient graphs in Refs. 1 and 2. This report discusses the cumulative detection probability (P_c) and contains curves of P_c as a function of normalized radar range.

II. EQUATION FOR CUMULATIVE DETECTION PROBABILITY

In discussing blip-scan ratio, it was convenient to normalize the range in terms of R_0 , the range for unity signal-to-noise ratio. In terms of radar parameters:

$$R_0^4 = \frac{\bar{P} G A \sigma T_d}{(4\pi)^2 k T_{\text{eff}} L} \quad (1)$$

assuming coherent integration for the time T_d or equivalently that the signal is passed through a matched filter. This form of the range equation can be derived from the more conventional form by substituting the following appropriate relations: for a continuous wave (cw) radar

$$\frac{\hat{p}}{\bar{P}} = 1 \approx B T_d$$

or for pulse radar

$$\frac{\hat{p} \text{ fr}}{\bar{P} B} \approx 1 \approx \frac{N_c}{T_d \text{ fr}}$$

In Refs. 1 and 2, blip-scan ratio is given for several different target scintillation models as a function of R/R_0 . Here we will denote blip-scan ratio by $p(R/R_0)$.

The cumulative detection probability is given in terms of this quantity by:

$$P_c(R, \Delta) = \frac{1}{\Delta} \int_R^{R+\Delta} dR' \left\{ 1 - \prod_{m=0}^{\ell} \left[1 - p\left(\frac{R' + m \Delta}{R_0}\right) \right] \right\} \quad (2)$$

The quantity Δ is the distance a target travels radially during a search frame. The quantity in large brackets in Eq. (2) is the probability that

a target which is scanned by the radar beam at a range R' , is detected at least once by this range. The integration, or averaging of this quantity over an interval Δ , accounts for the fact that the target could (with equal probability) be anywhere in the interval Δ when last traversed by the radar beam. The value for l is selected so that the blip-scan ratio is negligible, i.e., comparable to false alarm probability, at ranges in excess of $(R' + l\Delta)$.

For simplicity it is assumed that the signals received from a target during each scan are integrated coherently. Noting that:

$$\frac{G\omega}{4\pi} = \frac{\Delta}{V_c T_f} = \frac{T_d \Omega}{T_f \omega} = 1 \quad (3)$$

Eq. (1) can be rewritten in the following form:

$$R_o^4 = R_1^3 \Delta; R_1^3 = \frac{\bar{P} A \sigma}{4\pi k T_{eff} L \Omega V_c} Q_i^3 \quad (4)$$

where Q_i^3 is a correction factor explained in Section V and the Appendix. For the present, assume $Q_i = 1$. (This is equivalent to the assumption of coherent signal integration and a false alarm number of 10^6 for the work that follows.) In discussing cumulative detection probability it will be convenient to normalize the range in terms of R_1 , which is defined in Eq. (4). The interval Δ is retained as a parameter and can be varied to optimize the radar performance. All of the other independent radar and target parameters which influence the cumulative detection probability are contained in R_1 in their proper relationship.

For a given R_1 , i.e., for a fixed set of radar parameters ($\bar{P}, A, T_{eff}, L, \Omega$) and target characteristics (σ, V_c), increasing the interval Δ (or the frame time) increases the time for coherent signal integration and improves

the signal-to-noise ratio. This results in an increased blip-scan ratio. On the other hand, increasing Δ decreases the number of looks at a target and hence tends to decrease P_c . As might be expected, there is an optimum value of Δ and an optimum frame time for any assumed set of the other parameters.*

Setting $\frac{R}{R_1} = v$ and $\frac{\Delta}{R_1} = \delta$, Eq. (2) becomes:

$$P_c(v, \delta) = \frac{1}{\delta} \int_v^{v+\delta} dv' \left\{ 1 - \prod_{m=0}^{\ell} \left[1 - p(v' \delta^{-1/4} + m \delta^{3/4}) \right] \right\} \quad (5)$$

Assume one is interested in obtaining a given cumulative detection probability, say 90 per cent. From Eq. (5), the normalized range for 90 per cent P_c can be obtained as a function of δ and the optimum value of δ determined. This optimum value of δ is a function only of the target scintillation characteristics and the value of P_c which is specified. Since $\delta = \frac{\Delta}{R_1}$, it follows that Δ for optimum performance is proportional to R_1 , i.e., longer-range radars should use longer frame times (or, more exactly, larger Δ).

For an optimum frame time (or δ) the range at which a given P_c is obtained varies with the cube root of the power-aperture product. This is apparent from the fact that for an optimized system, i.e., optimum δ for the desired P_c , or more generally for any fixed value of δ , cumulative detection probability is a function only of v . Since v equals R/R_1 , the

* A similar problem was considered in Ref. 3, where post-detection integration rather than coherent integration was assumed to vary with scan interval. The assumption of varying coherent integration, which is made here, leads to a simpler formulation of the problem. The factor Q allows for the introduction of a specified amount of post-detection integration.

range for a given P_c is proportional to R_1 and hence to the cube root of the power-aperture product. This is different from the well known fourth-root relationship which is applicable when one is interested in blip-scan ratios.

It should also be noted, that a similar R^3 relation holds in other radar search problems, for instance, in the case of a fan-beam search radar such as BMEWS, if coherent integration is assumed. In the case of an airborne side-looking mapping radar the R^3 relation also holds. In each of these cases the R^3 relation holds because integration time increases with range.

III. SELECTION OF OPTIMUM FRAME TIME

Figure 1 gives the computed values of v (i.e., R/R_1) as a function of δ for several values of P_c . These curves were obtained by evaluating Eq. (5) numerically and they apply to the case of a non-fluctuating target. For example, if one is interested in a cumulative detection probability of 75 per cent, the optimum δ is 0.11. For this choice of δ , a 75 per cent cumulative detection probability is obtained at a range $R = .216 R_1$.

It can be seen from these curves that range (or R/R_1) is a slowly varying function of δ around the optimum and therefore that the choice of δ is not critical. The optimum value of δ decreases with increasing P_c . Frame time is related to δ by the simple expression:

$$T_f = \frac{\delta R_1}{V_c} \quad (6)$$

The optimum value of δ is given in Fig. 2 as a function of P_c , again for a non-fluctuating target. Figures 3 and 4 show how the optimum δ varies with P_c for two other target scintillation models as indicated in the figures.

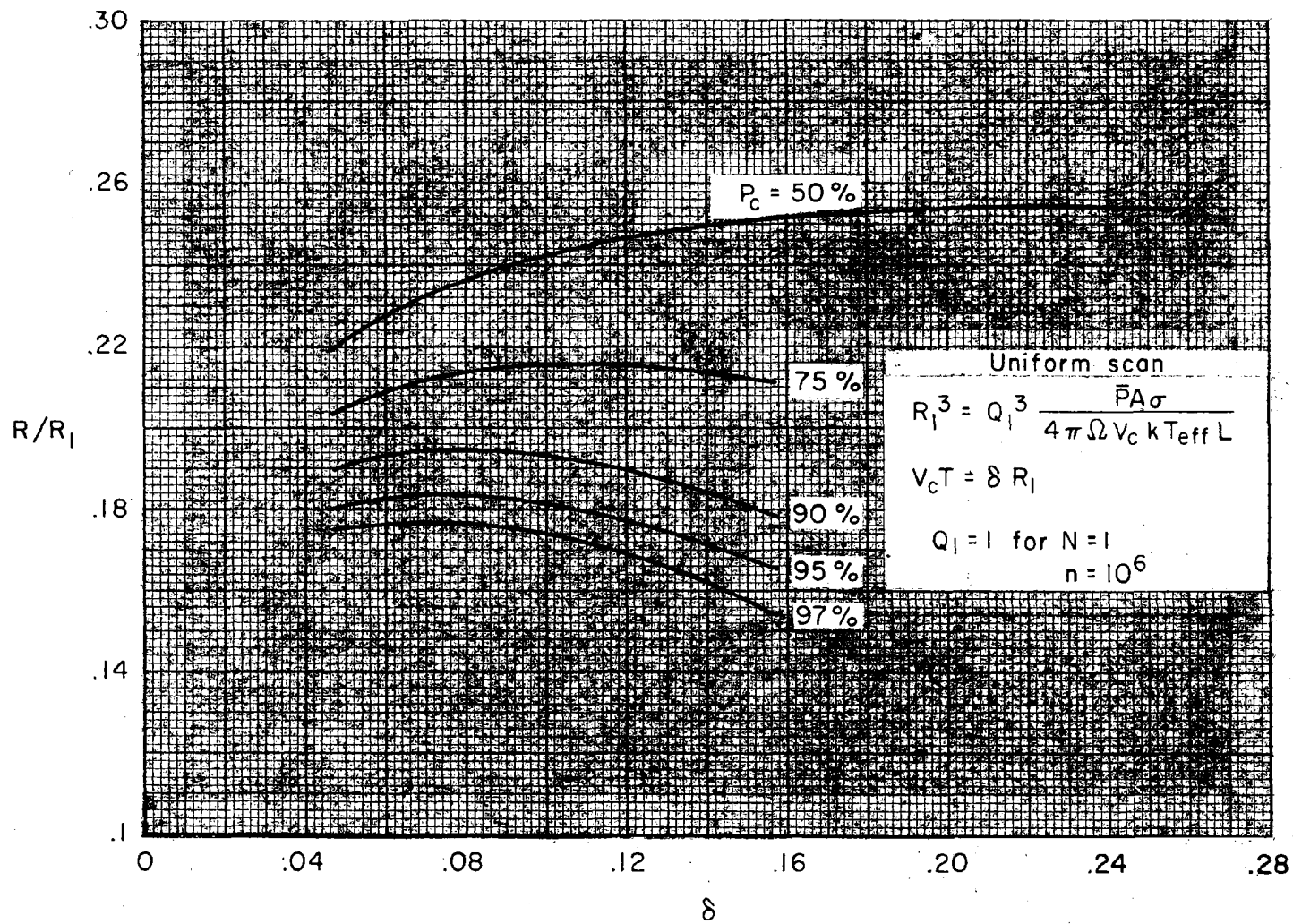


Fig. 1 — R/R_1 versus δ nonfluctuating target model.

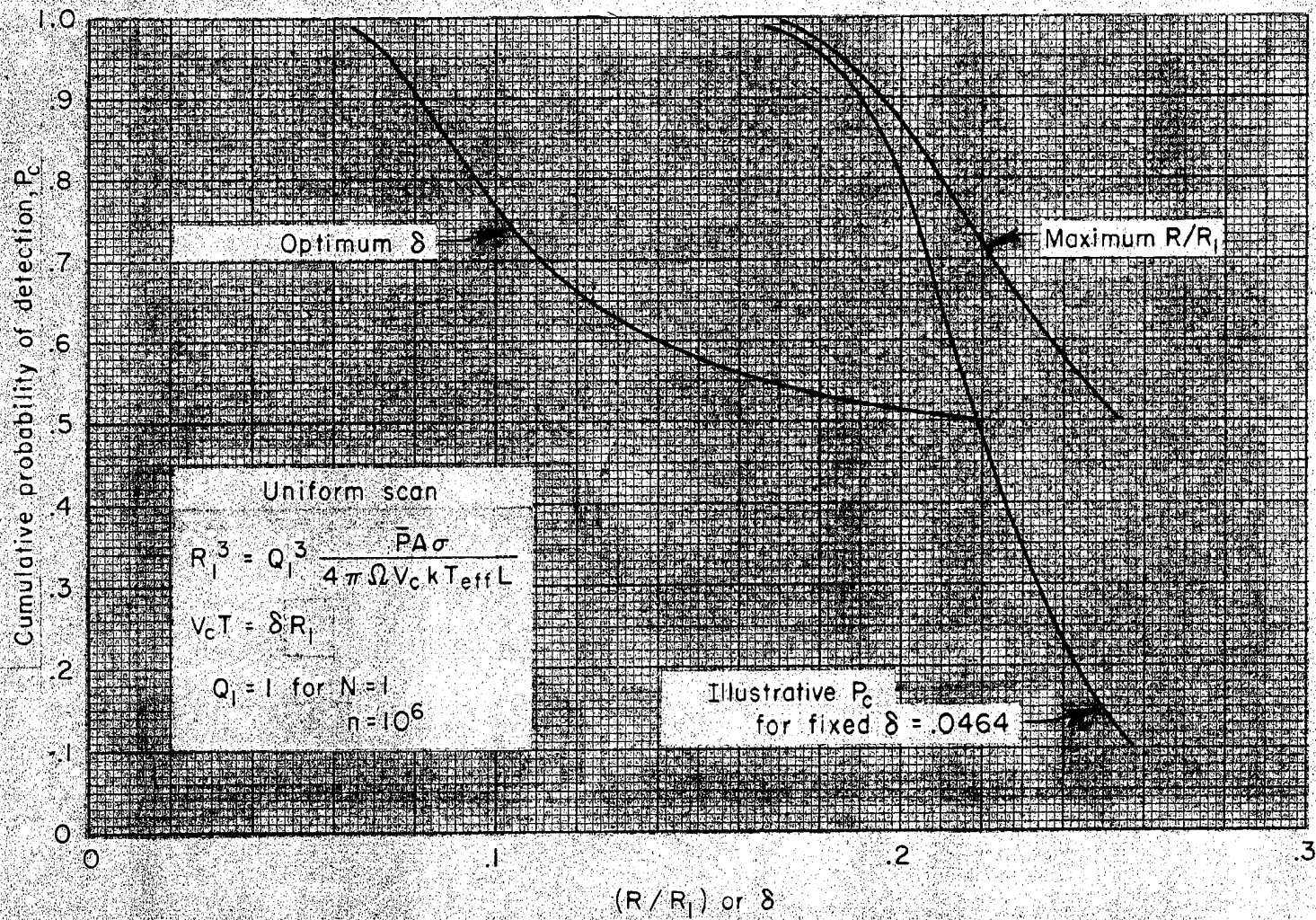


Fig. 2— Cumulative probability of detection by range R nonfluctuating target model

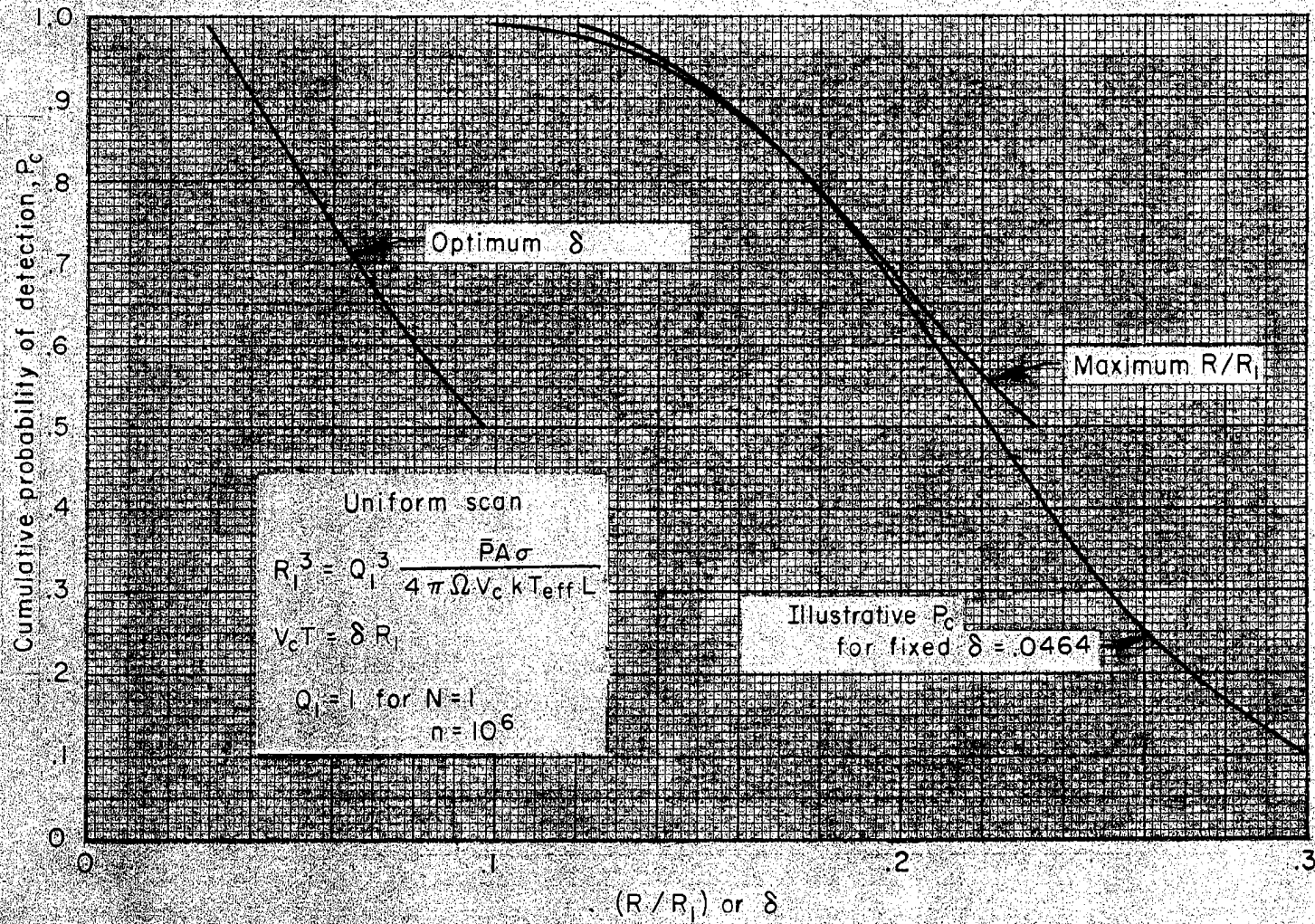


Fig. 3—Cumulative probability of detection by range R
 target fluctuating scan to scan $W(\sigma, \bar{\sigma}) = \frac{1}{\bar{\sigma}} e^{-\sigma/\bar{\sigma}}$ (Case I Swerling)

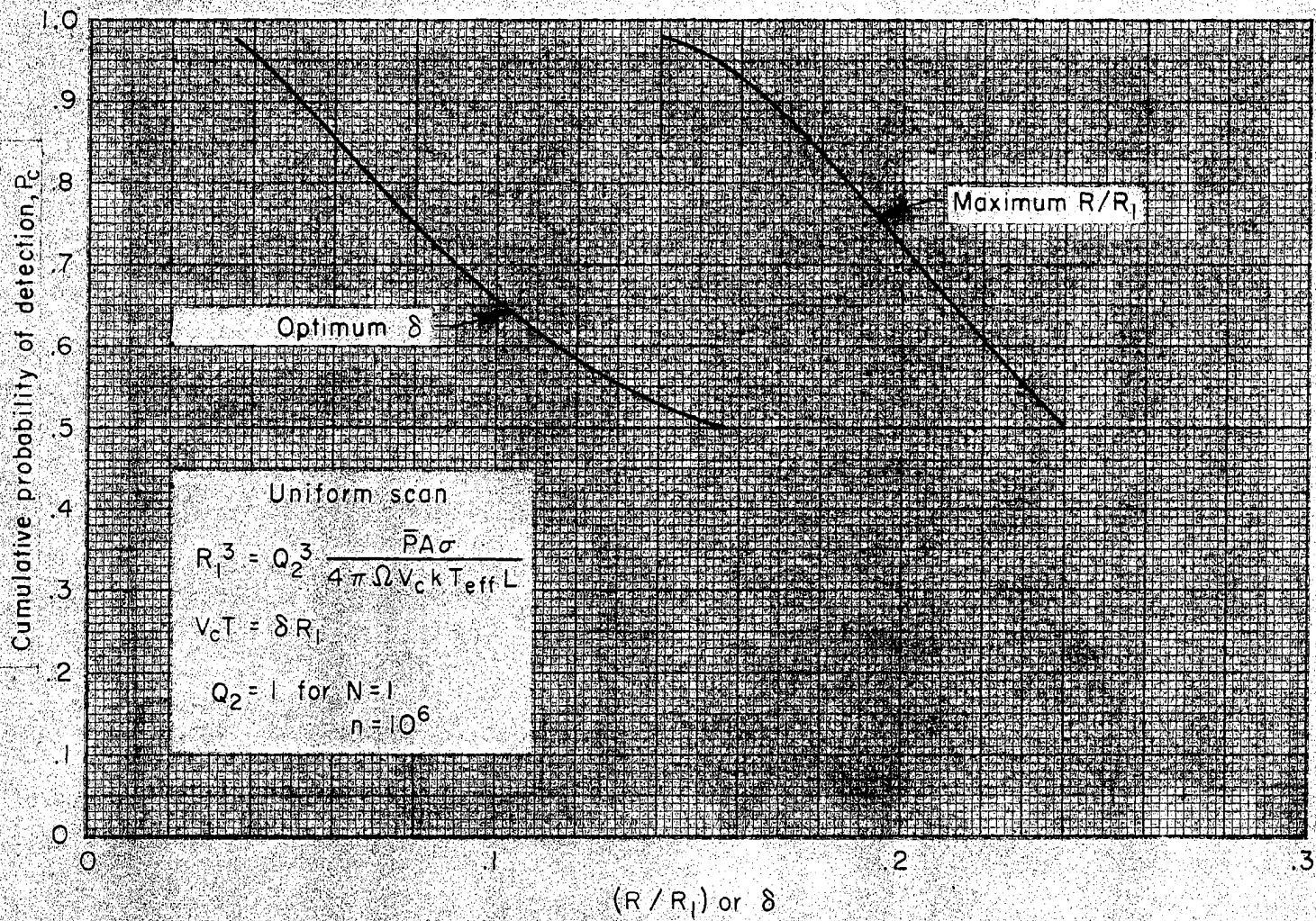


Fig. 4—Cumulative probability of detection by a range R target fluctuating scan to scan $W(\sigma, \bar{\sigma}) = \frac{4\sigma}{\bar{\sigma}^2} e^{-2\sigma/\bar{\sigma}}$ (Case III Swerling)

IV. CURVES OF CUMULATIVE DETECTION PROBABILITY VERSUS RANGE

Figures 2, 3, and 4 also show how cumulative detection probability varies with normalized range (R/R_1) for three target scintillation models. In each case the curve labeled maximum R/R_1 applies to the case where frame time (or δ) has been selected to maximize the range at which the given P_c is obtained. Thus δ varies with P_c or R/R_1 along these curves. For purposes of illustration, Figs. 2 and 3 also show P_c as a function of normalized range for a fixed value of δ ($\delta = .0464$).

To illustrate the use of these curves, assume one wishes to determine the range at which a given radar-target combination would yield an 80 per cent cumulative detection probability. First, R_1 can be calculated using Eq. (4), the radar parameters ($\bar{P}, A, T_{\text{eff}}, L$, and Ω), and the target parameters (σ and V_c). Assume the target is non-fluctuating so that Fig. 2 is applicable. For 80 per cent cumulative detection probability, the optimum value of δ is .094. The optimum frame time can be calculated using this value of δ and Eq. (6). The normalized range for 80 per cent P_c is $R/R_1 = 0.21$. Hence, the calculated range in this case is $0.21 R_1$.

The formulation of the search problem is quite general in that only the fundamental radar parameters are displayed. The beam width and frequency, for instance, do not appear in Eq. (4) for R_1 . Thus there is considerable latitude in radar design. For instance, the same aperture can be used for transmission and reception, or a smaller one could be chosen for transmission, in which case the A applies to the receiving aperture and multiple receiving beams must be used. A narrow beam or a

single large beam can be used to scan the search volume. In the latter case the scan time is to be interpreted as an integration time. The loss factor (L) in R_1 should, of course, include scan loss, field degradation, a factor for transmitting antenna inefficiency (power radiated into the sidelobes), plumbing loss, and other losses appropriate to the particular system.

Curves of averaged single-scan detection probability, i.e., average of blip-scan ratio over the interval Δ , have also been computed but are not shown here. The formulation of the problem in that case is very similar to the formulation for cumulative detection probability. Again the results can be presented conveniently in terms of v and δ . It is interesting that the averaged single-scan detection probability and the cumulative probability are almost identical for near-optimum values of δ in the case of a non-fluctuating target. Because of the steepness of the blip-scan curve in this case, the detection probability accumulated in earlier scans is of little significance. In the case of fluctuating targets the gain due to cumulation is quite significant.

V. NON-COHERENT SIGNAL INTEGRATION

The discussion thus far has assumed that the signal received from a target during each search frame is integrated coherently. Also, the curves assume $n = 10^6$, where n is the average number of noise samples between false alarms. If a different false alarm number is of interest ($n \neq 10^6$), or if the signal is integrated incoherently, a value of Q_1 different from unity should be used in Eq. (4) for computing R_1 . Aside from this difference, the procedures for determining optimum frame time and cumulative detection probability outlined above are still applicable. As explained in the Appendix, $N = 1$ in the Q_1 factor corresponds to coherent integration. If a value N_1 is assigned to N , this means that only $1/N_1$ of the original radar returns are integrated coherently, and that N_1 samples are integrated after envelope detection.

The quantity Q_1 should be used in the cases of a non-fluctuating target or a target with a Rayleigh amplitude distribution, i.e., Q_1 should be used in conjunction with Figs. 2 and 3. The quantity Q_2 should be used for the target scintillation model of Fig. 4. These quantities, Q_1 and Q_2 , are given in Figs. 5 and 6 for three different false alarm numbers, as a function of the number of samples integrated incoherently (N). The derivation of the Q_1 curves is discussed in the Appendix.

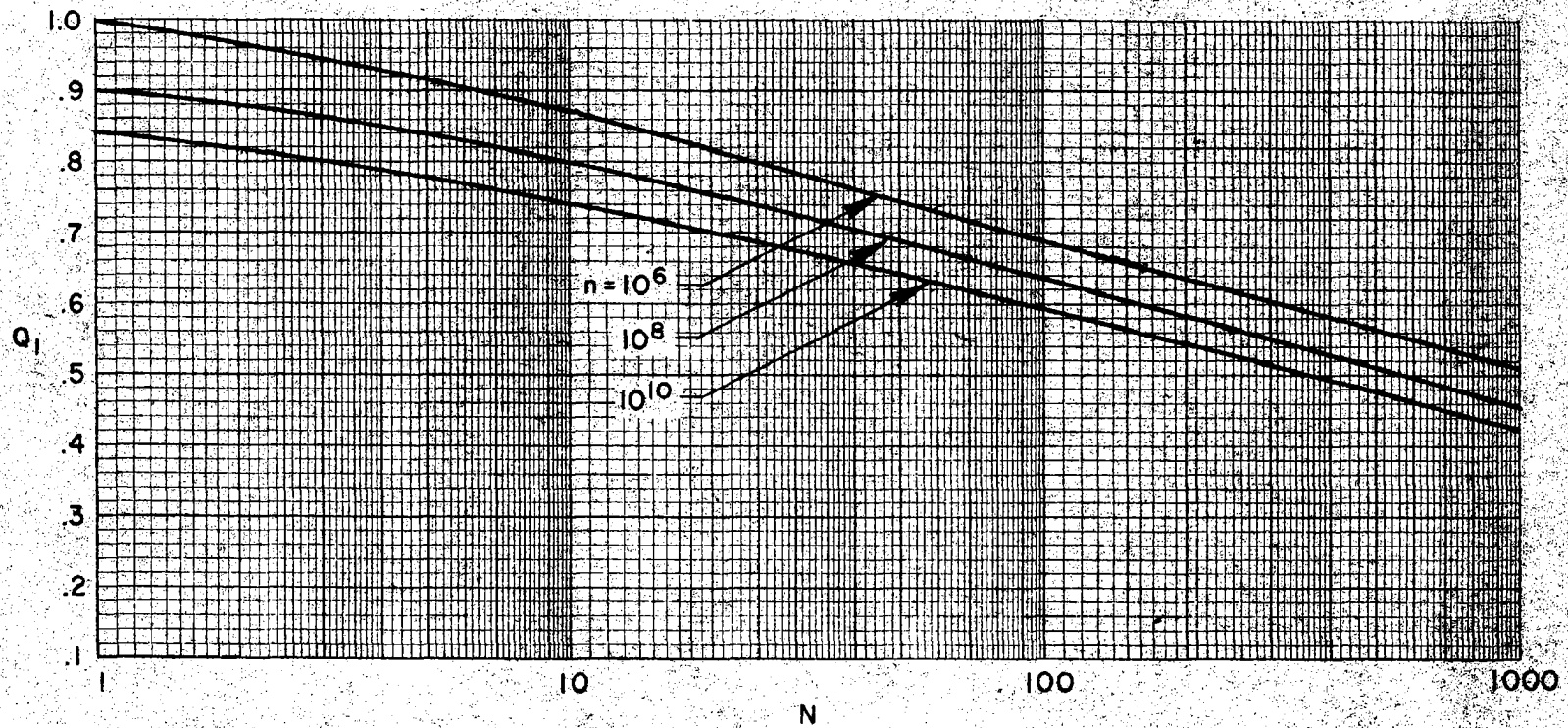


Fig. 5 — $Q_1(N, n)$

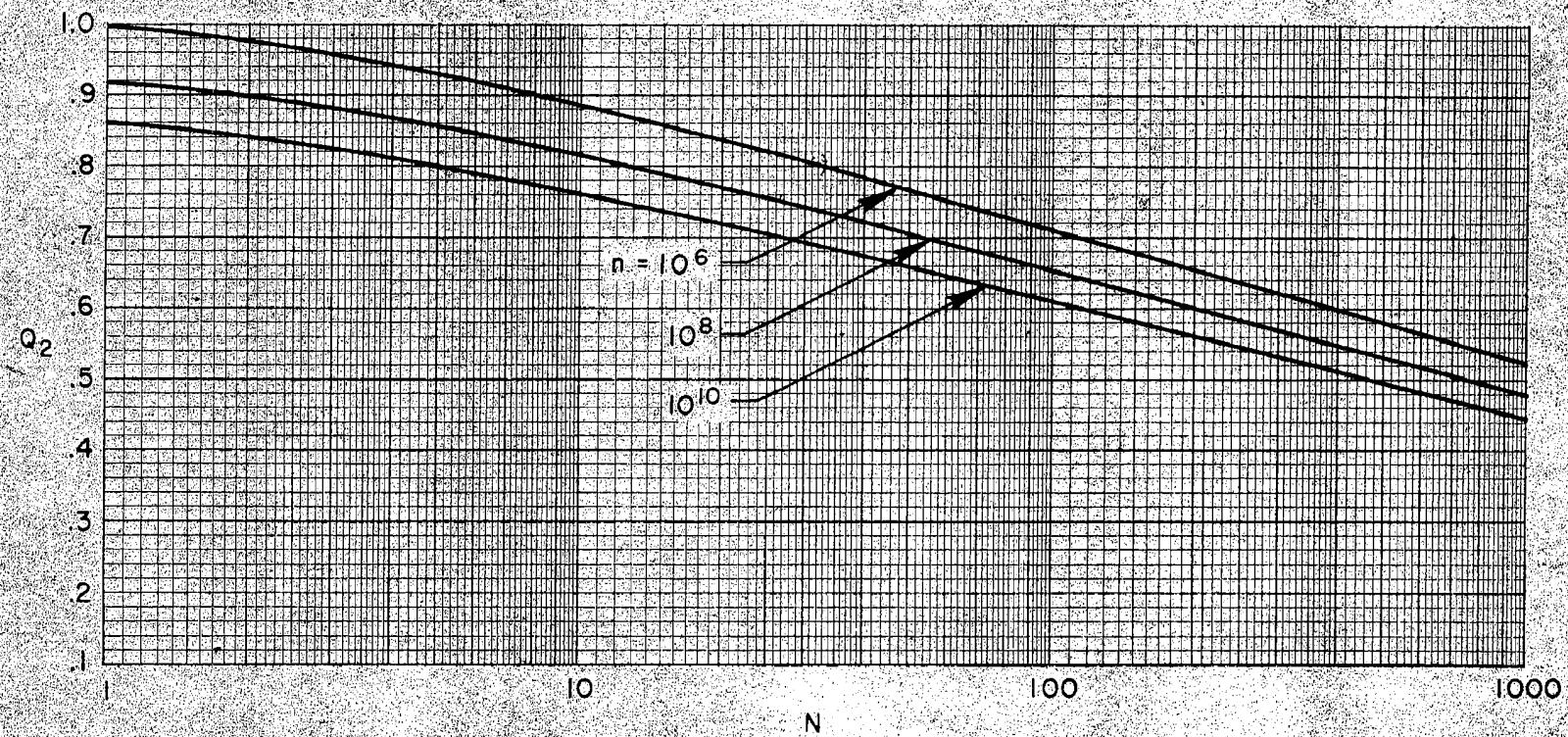


Fig. 6 — $Q_2(N, n)$

VI. CONCLUSIONS

The range at which a uniformly scanning radar achieves a given cumulative detection probability against an approaching target varies as the cube root of the radar's power-aperture product, assuming either optimum radar design (i.e., optimum scan interval and coherent signal integration) or a fixed deviation from optimum (e.g., scan interval equal to a constant times its optimum value in all cases).

Curves are given in Figs. 2 through 6 for computing radar range as a function of cumulative detection probability and for optimizing frame time. Three different target scintillation models were included, as indicated in Figs. 2 to 4. Since the work was done graphically the accuracy is not better than a few per cent. However, in most cases target echo area and field degradation are not known with sufficient accuracy to warrant more accurate range predictions. The accuracy is better for $n = 10^6$ and $N = 1$ than for other values, since an approximation is made to arrive at the other values.

The results are, of course, applicable to either mechanically scanned or phased-array radars. An interesting area for further analysis is the use of sequential detection techniques with phased-array radars in order to optimize cumulative detection probability.

Appendix

DERIVATION OF THE Q CORRECTION FOR NON-COHERENT INTEGRATION

In Ref. 2 it was shown that blip-scan ratio (p) can be expressed as a function of a single variable, u , to a good approximation in the case of a target fluctuating scan-to-scan with a Rayleigh amplitude distribution.

In this case:

$$u = \frac{R}{R_0} g(n, N) \quad (7)$$

where $g(n, N)$ is a correction factor. It is defined and plotted in Ref. 2. The blip-scan ratio for any arbitrary (n, N) can be obtained from the blip-scan curve for $(n = 10^6, N = 1)$ by multiplying the value of R_0 computed from Eq. (1) by:

$$\left(\frac{1}{N^{1/4}} \cdot \frac{g(10^6, 1)}{g(n, N)} \right) \quad (8)$$

For a given dwell time in Eq. (1), when the number of samples integrated incoherently increases from 1 to N , the amount of coherent integration decreases by the factor N . This accounts for the $N^{1/4}$ term above.

Since:

$$R_0^4 = R_1^3 \Delta \quad (9)$$

multiplying the value of R_0 by the factor of Eq. (8) is equivalent to multiplying R_1 by:

$$Q_1 = \left[\frac{g(10^6, 1)}{g(n, N)} \right]^{4/3} \cdot \frac{1}{N^{1/3}} \quad (10)$$

This is the quantity plotted in Fig. 5 and it is applicable to the target scintillation characteristic of Fig. 3. The quantity Q_1 can also be used

to a good approximation for non-fluctuating targets (Fig. 2) to extend the results to other values of N . The extension to different values of n does not hold to as good an approximation in this case but it is not far off for small changes in n . For example, one calculation for the case of a non-fluctuating target indicated ~5 per cent range error in using Q_1 to change n from 10^6 to 10^{10} . An identical argument, based on the scale factor $f(n, N)$ of Ref. 2, leads to a similar curve for Q_2 (Fig. 6) which is applicable for the target scintillation model of Fig. 4.

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