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RESEARCH MEMORANDUM

A METHOD FOR CHOOSING AMONG OPTIMUM STRATEGIES

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The purpose of this note is to set down for the record an observation made some months ago about a method of extending the max-min principle to choose (essentially uniquely) a preferred (in some sense) optimum strategy in discrete games whose solutions are not unique.

Basically the problem may be viewed as that of taking maximum advantage of an opponent's possible departures from optimum strategies, choosing from the set of optimum strategies. Observe, however, that it is impossible to take any advantage of an opponent's behavior as long as he uses only those pure strategies which can be represented with non-zero probability in an optimum mixture, since any of these pure strategies yields V, the value of the game, against any optimum strategy. Let us refer to such pure strategies as "admissible". These will be the only pure strategies which yield V against every optimum mixture.

Adopting the point of view of the maximizing player, the suggested procedure is to replace the original game matrix A by a derived matrix $\bar{A}$ obtained by (1) replacing the maximizing player's pure strategies by all the basic solutions for that player and (2) striking out the admissible pure strategies of the minimizing player. The solutions of $\bar{A}$ for the maximizing player will all yield solutions of A with the property that the minimum gain is maximized for any departure of the opponent from his set of admissible strategies. If the solution of $\bar{A}$ is still not unique the process may be repeated. Since at least one of II's strategies is admissible it is clear that this procedure ultimately terminates, either with a unique strategy for player I or with a situation in which player II's remaining strategies are all admissible. In the latter case, all the solutions for player I at that stage yield identical payoffs for all strategies of player II, hence may be considered equivalent. In either case, therefore, the result is unique, with respect to the set of payoffs.

Observe that if player I has a "uniformly best" solution, that is, a mixture which dominates all other solutions with respect to the payoff against the pure strategies of player II, then the procedure suggested must yield the uniformly best solution. Furthermore, the philosophical justification of the procedure, as an appeal to max-min in the absence of other information, seems quite reasonable.

As an example consider the game of "Morra", whose payoff matrix is given below.

Show		1			2			3		
	Guess	1	2	3	1	2	3	1	2	3
1	1	0	2	2	-3	0	0	-4	0	0
	2	-2	0	0	0	3	3	-4	0	0
	3	-2	0	0	-3	0	0	0	4	4
2	1	3	0	3	0	-4	0	0	-5	0
	2	0	-3	0	4	0	4	0	-5	0
	3	0	-3	0	0	-4	0	5	0	5
3	1	4	4	0	0	0	-5	0	0	-6
	2	0	0	-4	5	5	0	0	0	-6
	3	0	0	-4	0	0	-5	6	6	0

The basic strategies are

(1)	0	0	$\frac{5}{12}$	0	$\frac{4}{12}$	0	$\frac{3}{12}$	0	0
(2)	0	0	$\frac{16}{37}$	0	$\frac{12}{37}$	0	$\frac{9}{37}$	0	0
(3)	0	0	$\frac{20}{47}$	0	$\frac{15}{47}$	0	$\frac{12}{47}$	0	0
(4)	0	0	$\frac{25}{61}$	0	$\frac{20}{61}$	0	$\frac{16}{61}$	0	0

Replacing player I's pure strategies by the four basic solutions gives

Show	1			2			3		
Guess	1	2	3	1	2	3	1	2	3
(1)	$\frac{2}{12}$	0	0	$\frac{1}{12}$	0	$\frac{1}{12}$	0	0	$\frac{2}{12}$
(2)	$\frac{4}{37}$	0	0	0	0	$\frac{3}{37}$	0	$\frac{4}{37}$	$\frac{10}{37}$
(3)	$\frac{8}{47}$	$\frac{3}{47}$	0	0	0	0	0	$\frac{5}{47}$	$\frac{8}{47}$
(4)	$\frac{14}{61}$	$\frac{4}{61}$	0	$\frac{5}{61}$	0	0	0	0	$\frac{4}{61}$

The admissible strategies correspond to those columns which have all 0's. Striking these out, the solution of the reduced game is at first sight not unique for player I, but is unique for player II. Basic solutions for player I are (if there hasn't been a slip)

$$\text{a: } \left(\frac{120}{5\cdot 55}, 0, \frac{94}{5\cdot 55}, \frac{61}{5\cdot 55}\right) \text{ and } \text{b: } \left(\frac{12}{110}, \frac{37}{110}, 0, \frac{61}{110}\right),$$

which, however, both yield the same optimum mixture,

$$0 \quad 0 \quad \frac{23}{55} \;\Big|\; 0 \quad \frac{18}{55} \quad 0 \;\Big|\; \frac{14}{55} \quad 0 \quad 0 \; .$$

So that this may be checked, we give player II's solution for the reduced game, $\left(0, \frac{61}{110}, 0, \frac{48}{110}, \frac{1}{110}, 0\right)$, striking out the three columns which all have zeros. The value is $\frac{2}{55}$, which will be guaranteed to player I if player II ever departs from an optimum strategy in the original game.

It is not clear how this should generalize to continuous games. One approach might be to maximize the minimum gradient for departures from player II's admissible set of strategies.