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RESEARCH MEMORANDUM

"BEST" STRATEGIES

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On occasion some of us here have considered the possibility of picking out special strategies, called "best" strategies, which have the special feature that, in addition to being optimum in the usual sense, they take advantage of the mistakes of one's opponent. I show here that such best strategies always exist, in the case of finite games, and I raise some questions regarding the nature of the set of all best strategies.

If Γ is a zero-sum two-person game then by

$$E_{\Gamma}(\xi, y)$$

we shall mean the expectation of the first player when he uses the mixed strategy ξ, and the second player uses the pure strategy y; when there is no danger of ambiguity, we shall omit the "Γ", writing merely

$$E(\xi, y).$$

"We say that a strategy ξ_1 dominates a strategy ξ_2 if, for all y,

$$E(\xi_1, y) \geq E(\xi_2, y).$$

with the inequality holding for at least one y. ξ is called a best strategy if it is optimal, and is not dominated by any other strategy. A best strategy is called uniformly best, if it dominates every other strategy."

It is easy to give an infinite game for which the set of best strategies is empty. Thus suppose, for example, that the first player can choose any positive integer, while the second player can choose either 1 or 2, and that the payoff function H (to the first player) is defined as follows:

$$H(i, 1) = 0 \qquad \text{for all } i$$

$$H(i, 2) = \frac{i-1}{i} \qquad \text{for all } i;$$

thus the (infinite) payoff matrix for this game is as follows:

$$\begin{matrix} 0 & 0 \\ 0 & \frac{1}{2} \\ 0 & \frac{2}{3} \\ 0 & \frac{3}{4} \\ \vdots & \vdots \end{matrix}$$

The matrix has a saddle-point at $\langle n, 1 \rangle$ for every n and the second player has a unique optimum strategy -- namely, he should always play the first column. On the other hand, every strategy is optimum for the first player. Moreover, every strategy for the first player is dominated by some other: if $\xi = \langle \xi_1, \xi_2, \ldots \rangle$ is any strategy, then $\xi' = \langle 0, \xi_1, \xi_2, \ldots \rangle$ is a strategy which dominates it, for

$$E(\xi', 1) = E(\xi, 1)$$

and

$$E(\xi', 2) > E(\xi, 2).$$

Thus there is no best strategy for the first player.

In contradistinction, for the case of finite games we have the following:

Theorem 1. The class of best strategies for a finite game is not empty.

Proof. Let

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

be the payoff matrix of a finite game.

Let B_o be the set of optimum mixed strategies for the first player. As is well-known, B_o is a bounded closed set in m-dimensional space. Let the function f_1 be defined as follows:

$$(1) \qquad f_1(x_1, \ldots, x_m) = \sum_{i=1}^{m} a_{i1}x_i .$$

Since f_1 is a continuous function, and B_o is bounded and closed, there exists a point $\langle x_1^{(1)}, \ldots, x_m^{(1)} \rangle$ in B_o such that

$$(2) \qquad f_1(x_1^{(1)}, \ldots, x_m^{(1)}) = \max_{\langle x_1, \ldots, x_m \rangle \in B_o} f_1(x_1, \ldots, x_m).$$

Now let B_1 be the set of all points $\langle x_1, \ldots, x_m \rangle$ of B_o such that

$$(3) \qquad f_1(x_1, \ldots, x_m) = f_1(x_1^{(1)}, \ldots, x_m^{(1)}).$$

From (1) and (3) we conclude that B_1 is bounded and closed. From (3) we see that B_1 is not empty.

Now suppose that $B_1, \ldots, B_k$ (with $k < n$) have been defined, and are known to be non-empty, bounded, and closed. We set

$$f_{k+1}(x_1, \ldots, x_m) = \sum_{i=1}^{m} a_{i,k+1}x_i.$$

Since B_k is bounded and closed, and f_{k+1} is continuous, f_{k+1} assumes its maximum at some point $\langle x_1^{(k+1)}, \ldots, x_m^{(k+1)} \rangle$ of B_k: i.e., we have

$$f_{k+1}(x_1^{(k+1)}, \ldots, x_m^{(k+1)}) = \max_{\langle x_1, \ldots, x_m \rangle \in B_k} f_{k+1}(x_1, \ldots, x_m).$$

We denote by B_{k+1} the set of all points $\langle x_1, \ldots, x_m \rangle$ of B_k such that

$$f_{k+1}(x_1, \ldots, x_m) = f_{k+1}(x_1^{(k+1)}, \ldots, x_m^{(k+1)}),$$

and it is readily seen that B_{k+1} is non-empty, bounded, and closed.

Thus we have a sequence B_o, B_1, ..., B_n of sets of points of n-space which satisfy the following conditions:

(1) B_o is the set of optimal strategies for the first player;

(2) for $k = 1, \ldots, n$, B_k is the set of points of B_{k-1} at which the form

$$\sum_{i=1}^{m} a_{i,k} x_i$$

assumes its maximum.

(3) B_n is not empty.

It is now easily seen that every member of B_n is a best strategy. For suppose that $\langle u_1, \ldots, u_m \rangle \in B_n$ and that $\langle v_1, \ldots, v_m \rangle$ is an optimum strategy such that, for $k = 1, \ldots, n$,

$$\sum_{i=1}^{m} a_{ik} v_i \geq \sum_{i=1}^{m} a_{ik} u_i .$$

Since $B_n \subseteq B_{n-1} \subseteq \ldots \subseteq B_1 \subseteq B_o$, we see that $\langle u_1, \ldots, u_m \rangle \in B_1$, and hence that

$$\sum_{i=1}^{m} a_{i1} u_i = \max_{\langle x_1, \ldots, x_m \rangle \in B_o} \sum_{i=1}^{m} a_{i1} x_i ,$$

so that

$$\sum_{i=1}^{m} a_{i1} v_i = \sum_{i=1}^{m} a_{i1} u_i .$$

We conclude that $\langle v_1, \ldots, v_m \rangle \in B_1$;

then since

$$\sum_{i=1}^{m} a_{i2} v_i \geq \sum_{i=1}^{m} a_{i2} u_i,$$

and

$$\sum_{i=1}^{m} a_{i2} u_i = \max_{\langle x_1, \ldots, x_m \rangle \in B_1} \sum_{i=1}^{n} a_{i2} x_i,$$

we see that

$$\sum_{i=1}^{m} a_{i2} v_i = \sum_{i=1}^{m} a_{i2} u_i,$$

and hence that $\langle v_1, \ldots, v_m \rangle \in B_2$. Continuing in this way, we see that $\langle v_1, \ldots, v_m \rangle \in B_k$, for $k = 1, \ldots, n$, and that

$$\sum_{i=1}^{m} a_{ik} v_i = \sum_{i=1}^{m} a_{ik} u_k \qquad \text{for } k = 1, \ldots, n.$$

Thus $\langle u_1, \ldots, u_m \rangle$ is not dominated by $\langle v_1, \ldots, v_m \rangle$, as was to be shown.

Remark. It would apparently be of interest to study the geometrical properties of the set of all best strategies for a finite game.

It can be shown that every optimum strategy which is not best is dominated by a best strategy. (It would be desirable to give a simple proof of this; the only proof I have found so far is rather tedious.)

Some interesting questions regarding the set B of all best strategies are the following:

Is B a closed subset of n-space?

Under what conditions is B finite?

Is B connected?

Is B polygonal?

Can it be shown that B is non-empty for the case of a continuous game with a continuous payoff function?

There are also questions regarding uniformly best strategies. What, for instance, are necessary and sufficient conditions that a finite game have a uniformly best strategy?

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