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RESEARCH MEMORANDUM

SOME REMARKS ON BEST STRATEGIES

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In [1]* McKinsey defined a best strategy and raised some questions concerning the set of all such strategies. Brown has also discussed this question from a slightly different aspect in [2] . Recorded here are some comments upon the geometrical significance ** of best strategies.

According to the definition of best strategy given in [1] there is either a unique (uniformly) best strategy or more than one best strategy, and hence, infinitely many. The technique advocated in [2] for choosing one of the optimum strategies always yields a best strategy in the sense of [1] . A uniformly best strategy in the sense of [1] is always found by the technique described in [2] but it is actually unnecessary to use the suggested procedure for finding it since it is one of the extreme points (or basic solutions) of the set of optimal strategies.

The geometrical significance of the set of best strategies leads this writer to offer the following conjectures in answer to the questions asked at the end of [1] with reference to the set B (of best strategies). The image of the set B (obtained by applying the linear transformation represented by the payoff matrix to the set of optimal mixed strategies) is a closed connected, polygonal subset of n-space which is finite (consisting, in fact, of a unique element) only in case there exists a uniformly best strategy. A consideration of the geometry of the situation and the theory presented in [3] is the basis of these conjectures.

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* The numbers in square brackets refer to papers listed at the end of research memorandum.

** The reader will find the details of the geometry in [4] by L. S. Shapley.

Let S_k be the k-dimensional vector space of vectors $\vec{\alpha}$, where (the components) $\alpha_\nu \geq 0$ for all ν and $\sum_{\nu=1} \alpha_\nu = 1$. The possible mixed strategies $\vec{\xi}$ for player I form the convex polyhedron S_m in m-space. The set B_o)of McKinsey) is also a convex polyhedron, contained in S_m, spanned by extreme points of the set of optimal mixed strategies, say $\vec{\xi}^*$. Apply the linear transformation represented by the payoff matrix A to the regions S_m and B_o. Write

$$\vec{L} = \vec{\xi}'A,$$

$$\vec{K} = \vec{\xi}^{*'}A,$$

where the prime indicates the transpose of the column matrix (vector) $\vec{\xi}$. Thus the $\vec{L}$ span a convex polyhedron, say C_1, in n-space and the $\vec{K}$ span a convex polyhedron, say C_o, in n-space. The vertices of C_1 are just the rows of A and the vertices of C_o are those $\vec{L}$ for which

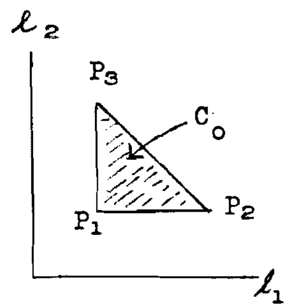
$$\min_j \ell_j = \min_j f_j \text{ (the } f_j \text{ of McKinsey)}$$

is a maximum, where $\vec{L}$ has components ℓ_j, $j = 1, 2, \ldots, n$. (The vectors $\vec{K}$ span this set.) Let $M_j(z)$ be that half of n-space for which $\ell_j \geq z$, where z is a real positive number. Now let z decrease from $+\infty$ to v, the value of the game represented by A, and consider the intersection of the produce space

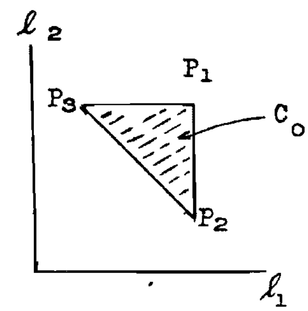
$$M = \prod_{j=1}^{n} M_j(z)$$

with region C_1. The intersection is the region C_o. Now it is easy to geometrically characterize* the best strategies and, if it exists, the uniformly best strategy. The former constitute the "upper" boundary of the region of contact. Consider two simple examples of regions of contact illustrated below. (The fact that such regions of contact are not possible except as degenerate cases of a more complicated situations is not pertinent to this discussion.)

I.



II.



Denote the vectors to the points P_1, P_2, and P_3 by $\overrightarrow{P_1}$, $\overrightarrow{P_2}$, and $\overrightarrow{P_3}$, respectively. In example I, any vector of the form

$$\overrightarrow{P_4} = \alpha \overrightarrow{P_3} + (1-\alpha) \overrightarrow{P_2}, \quad 0 \leq \alpha \leq 1,$$

dominates any other vector $\overrightarrow{P} \neq \overrightarrow{P_4}$ in the set C_o, spanned by $\overrightarrow{P_1}$, $\overrightarrow{P_2}$, and $\overrightarrow{P_3}$. So there would be infinitely many best strategies $\overrightarrow{P_4}$ (in the sense of McKinsey) and no uniformly best strategy. By taking $\alpha = \frac{1}{2}$ one obtains the strategy indicated by the Brown technique in [2] . In example II there would be a unique (uniformly) best strategy $\overrightarrow{P_1}$ (in the

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* I am grateful to H. F. Bohnenblust for a discussion of this matter and some very helpful suggestions.

sense of either [1] or [2]). Notice that it is an extreme point (represented by a basic solution to the game). Denote any extreme point of the set of optimal mixed strategies by $\overrightarrow{K}^{(\nu)}$ and consider the (strategy) vector

$$\vec{K}^* = \langle \max k_j^{(\nu)} \rangle , \quad j = 1, 2, \ldots, n.$$

If $\overrightarrow{K}^{*} \in C_o$ then it is the uniformly best strategy by either criterion and since it is an extreme point, the fact that it is uniformly best will be obvious by inspection. On the other hand, if there is no uniformly best strategy, Brown's technique would lead to a strategy (or possibly a set of strategies) which would be contained among the set of McKinsey's best strategies. We conjecture that if the Brownian technique leads to a unique result (which is not a uniformly best strategy) then it must be the intersection of the vector $\overrightarrow{K}^{*}$ ($\notin C_o$ in this case) and the "upper" boundary of C_o. In example I, this is the point characterized by the vector

$$\overrightarrow{P_4} = \frac{1}{2}\overrightarrow{P_2} + \frac{1}{2}\overrightarrow{P_3} \quad .$$

Consider the game with the payoff matrix

$$\begin{pmatrix} 46 & 22 & 39 & 27 \\ 10 & 34 & 39 & 27 \\ 55 & 31 & 48 & 24 \\ 19 & 43 & 12 & 36 \end{pmatrix}$$

The basic solutions for player I are

$$X^{(1)\prime} = \langle \frac{7}{12}, \frac{1}{12}, \quad 0, \frac{4}{12} \rangle ,$$

$$X^{(2)\prime} = \langle \frac{17}{36}, \frac{7}{36} , \quad 0, \frac{12}{36} \rangle ,$$

$$X^{(3)\prime} = \langle \ 0, \frac{28}{144}, \frac{51}{144}, \frac{65}{144} \rangle ,$$

$$X^{(4)\prime} = \langle \frac{28}{48}, \quad 0, \frac{3}{48}, \frac{17}{48} \rangle ,$$

$$X^{(5)\prime} = \langle \ 0, \quad 0, \quad \frac{1}{2}, \quad \frac{1}{2} \rangle ,$$

and the (unique) solution for player II is

$$Y = \langle 0, 0, \frac{1}{4}, \frac{3}{4} \rangle .$$

The value of the game is $v = 30$ and the simple computation of

$$K^{(v)} = X^{(v)\prime}A, \qquad v = 1, 2, \ldots, 5,$$

shows that

$$X^{(5)\prime} = \langle 0, 0, \frac{1}{2}, \frac{1}{2} \rangle$$

is the uniformly best strategy. On the other hand, the game represented by the payoff matrix

$$\begin{pmatrix} 4 & 0 & 2 & 1 \\ 0 & 4 & 1 & 2 \\ 1 & -1 & 3 & 0 \\ -1 & 1 & 0 & 3 \\ -2 & -2 & 2 & 2 \end{pmatrix} ,$$

with solutions

$$X = \langle \frac{4}{9}, \frac{4}{9}, 0, 0, \frac{1}{9} \rangle ,$$

$$Y^{(1)} = \langle \frac{3}{90}, \frac{7}{90}, \frac{48}{90}, \frac{32}{90} \rangle ,$$

$$Y^{(2)} = \langle \frac{7}{90}, \frac{3}{90}, \frac{32}{90}, \frac{48}{90} \rangle ,$$

and value $v = \frac{14}{9}$ has no uniformly best strategy for player II. In the sense of McKinsey any strategy of the form

$$Y^{(3)} = \alpha Y^{(1)} + (1-\alpha) Y^{(2)}, \quad 0 \leq \alpha \leq 1,$$

(and hence any one of the entire set of optimal strategies for player II) is a best strategy. Brown's preferred strategy is

$$Y^{(4)} = \frac{1}{2} Y^{(1)} + \frac{1}{2} Y^{(2)} .$$

BIBLIOGRAPHY

1. J. C. C. McKinsey, "Best" Strategies, RM-386.
2. G. W. Brown, A Method for Choosing Among Optimum Strategies, RM-376.
3. L. S. Shapley, S. Karlin, and H. F. Bohnenblust, Solutions of Discrete, Two-Person Games, P-57.
4. L. S. Shapley, A Geometric Conceptualization of Optimal Strategies in the Finite, Two-Person, Zero-Sum Game, RM-36.

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