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A STATISTICAL THEORY OF TARGET DETECTION
BY PULSED RADAR: MATHEMATICAL APPENDIX

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SUMMARY

In a previous report⁽²⁸⁾ a statistical theory of radar detection was presented in outline form. The mathematical details were omitted, in order that the main ideas and results might be made available as soon as possible.

This report contains the mathematics that led to the results presented in Ref.28.

In addition, several subjects are briefly discussed that were not covered in Ref.28. These are collapsing loss, antenna beam shape loss, the effect of signal injection, limiting loss, and moving target indication.

SYMBOLS

a	= amplitude of sine wave relative to R.M.S. noise level
α^{**}	= one of the independent variables in the function $Q(\alpha, \beta)$
α_i	= i^{th} central standard moment
β^{**}	= one of the independent variables in the function $Q(\alpha, \beta)$
B	= half-power antenna beamwidth
c_i	= coefficient in the Gram-Charlier series
C^*	= characteristic function
δ_{ij}	= delta function
e	= base of natural logarithms
f	= frequency
${}_1F_1$	= confluent hypergeometric function
F^*	= Campbell and Foster notation for characteristic function
G^*	= Campbell and Foster notation for anticharacteristic function
Γ	= the gamma function
Γ_N	= probability that the sum of N noise variates will exceed the bias level
H_i	= i^{th} Hermite polynomial
i	= index, subscript, or $\sqrt{-1}$
I	= incomplete gamma function as defined by Pearson ⁽⁸⁾
I_n	= modified Bessel function of the first kind
J_ν	= Bessel function of the first kind
κ_i	= i^{th} cumulant
K_i^{**}	= standard i^{th} cumulant, or sometimes a modified Bessel function of the second kind
L_i	= integration loss
L_c	= collapsing loss
L_i^α	= generalized Laguerre polynomial
M	= number of excess noise variates integrated with N signal plus noise variates
n	= false alarm number
n'	= n/N
N	= number of variates integrated

*This symbol has a different meaning in RA-15061.

**This symbol is used in more than one sense in various places, but other meanings should be obvious.

p^*	= $i\omega = \omega\sqrt{-1}$
P	= sine wave amplitude
P^{**}	= probability
P_0	= probability that noise will exceed the bias level at least once within false alarm time
P_N	= probability that the sum of N variates of signal plus noise will exceed the bias level
ϕ^i	= i^{th} derivative of the error function
ψ_0	= R.M.S. noise level
$Q(\alpha, \beta)$	= modified Lommel's function
R^{**}	= envelope amplitude or radar range in R/R_0
R_0	= idealized radar range
ρ	= collapsing ratio, ratio of total number of variates integrated to those containing signal
s	= cathode ray writing speed
σ^*	= standard deviation
T^*	= $(y-\bar{y})/\sigma$, semi-independent variable in Gram-Charlier series
T_i^*	= incomplete Toronto function
μ_i	= i^{th} moment about the mean
U_i	= Lommel's function
v^*	= normalized envelope amplitude
ν_i	= i^{th} moment
$w(f)$	= power spectrum
ω	= $2\pi f$
x	= power signal-to-noise ratio
y^*	= normalized detector output
Y^{**}	= integrator output for the sum of N variates
Y_b	= bias level

*This symbol has a different meaning in RA-15061.

**This symbol is used in more than one sense in various places, but other meanings should be obvious.

A STATISTICAL THEORY OF TARGET DETECTION BY PULSED RADAR: MATHEMATICAL APPENDIX

BASIC FORMULAE RELATING TO THERMAL NOISE

Both the thermal noise voltage across a resistor and the noise voltage due to the shot effect in a vacuum tube approach a normal distribution when the number of electrons involved per second in the processes tends toward infinity. In practice, it may usually be assumed that the total noise voltage between any two points due to any combination of thermal, shot, and cosmic noise sources can be represented by the distribution function

$$dP = \frac{1}{\sqrt{2\pi\psi_0}} e^{-\frac{V^2}{2\psi_0}} dV \quad (1)$$

where ψ_0 is the mean square value of the noise voltage⁽¹⁸⁾. This distribution is valid provided all elements involved in the composition of the total noise voltage have been linear.

If such noise is now passed through a linear filter of center frequency f_m , having a pass band which is narrow compared to f_m , the output will have an envelope, which has a probability density function

$$dP = \frac{R}{\psi_0} e^{-\frac{R^2}{2\psi_0}} dR \quad (2)$$

For references see page 111.

where R is the amplitude of the envelope and ψ_0 is the mean square noise voltage, given by

$$\psi_0 = \int_0^{\infty} w(f) df \quad (3)$$

$w(f)$ is the so-called power spectrum of the filter and is simply the square of the absolute value of the amplitude transfer function of the filter.

If the input to a filter consists of a sine wave of frequency f_n , as well as noise, then the probability density function of the output envelope is*

$$\begin{aligned} dP &= \frac{R}{\psi_0} e^{-\frac{R^2+P^2}{2\psi_0}} I_0\left(\frac{RP}{\psi_0}\right) dR, & R > 0 \\ dP &= 0 & R < 0 \end{aligned} \quad (4)$$

where P is the amplitude that the sine wave would have at the output of the filter in the absence of noise, and I_0 is a modified Bessel function of the first kind (see footnote, page 13).

The envelope of the output has a correlation time which is approximately equal to the reciprocal of the bandwidth of the filter. In simple language, it is improbable that the envelope will change by an appreciable percentage in times much less than the correlation time, but it is quite probable that it will change by a goodly percentage in times large compared with the correlation time. It is probably a good approximation to assume that values of the envelope $1/\Delta f$ seconds apart are independent, where Δf is the bandwidth of the filter. By assuming such a discrete process it is possible to materially simplify calculations which would be very tedious if exact integration processes were used, while at the same time sufficient accuracy is obtained for most practical purposes.

A further justification for this assumption in the pulsed case shows in the results. Changing the factor $1/\Delta f$ to $k/\Delta f$ for the correlation time has only the effect of changing the false alarm time by the factor k . The probability of detection turns out to be a very insensitive function of the false alarm time, so that if k is any factor of the order of magnitude of unity, the results are affected to a negligible extent.

* It is of some interest to note that the same form of distribution function occurs in other problems. For instance, if ψ_0 represents the mean square velocity of a gas due to ordinary turbulence, and P represents the translational velocity of the whole mass of gas relative to some fixed reference, then the density function of Eq.(4) gives the probability that the total vector velocity at any point in the gas will have a magnitude between R and $R + dR$ (24).

The same density function also represents the probability that a bomb will hit at a distance between R and $R + dR$ from a given point if it is initially aimed at a point whose distance from the given point is P . The mean square aiming error is represented by ψ_0 , the distribution being assumed Gaussian (47).

DEFINITION AND EFFECT OF DETECTION

A detector is defined as any device whose instantaneous output is a function of the envelope of the input wave only. Thus

$$y = f\left(\frac{R}{\sqrt{\psi_0}}\right) = F(v) \quad (5)$$

where y is the output of the detector and v is the normalized amplitude of the envelope.

If $P/\sqrt{\psi_0}$ is replaced by a , Eq.(4) may be written

$$dP = v e^{-\frac{v^2+a^2}{2}} I_0(av) dv, \quad v > 0 \quad (6)$$

$$dP = 0 \quad v < 0$$

$$\text{Eq.(5) solved for } v \text{ is } v = g(y) \quad (7)$$

If v is eliminated from (6) and (7), an equation of the form

$$dP = f(a, y) dy \quad (8)$$

is obtained for the probability density for the normalized voltage at the output of the detector which has the characteristics given by Eq.(5). For example, if $y = v^2/2$, then Eq.(8) becomes

$$dP = e^{-y-x} I_0(2\sqrt{xy}) dy, \quad y > 0 \quad (9)$$

$$dP = 0 \quad y < 0$$

where $a^2/2$ has been replaced by x . The quantity x may be identified with the power signal-to-noise ratio, commonly used in radar literature.

EFFECT OF VIDEO AMPLIFIERS

Since a complete radio receiver usually has one or more stages of video amplification following the detector, it would seem that one would want to calculate the probability density function for signal-plus-noise at the output of the video

amplifier. This can be done theoretically, as has been shown in an excellent paper by Kac⁽²⁵⁾, but the mathematical labor is great. On the other hand, it has been shown experimentally⁽³⁰⁾ that the signal threshold is practically unaffected by the video bandwidth until it becomes less than about 1/2 of the IF bandwidth. Since video bandwidths less than 1/2 of the IF bandwidth are quite uncommon in practice, it appears best for the sake of simplicity to assume the video bandwidth infinite in all the work which follows.

When the results have been computed, assuming an infinite bandwidth, it will be possible to modify them in an approximate manner so that they become valid for any video bandwidth. This is explained on page 59, under the title "Collapsing Loss".

PROBABILITY OF DETECTION WITH NO INTEGRATION

The calculations necessary to determine the probability of detection when exactly one correlation interval is available are quite simple compared with the case where the output over many correlation intervals is available, and hence the former case is taken up first. In a pulsed system this corresponds to using a single pulse, while in a c-w system it is equivalent to observing the output for a time $t = 1/\Delta f$, where Δf is the over-all effective bandwidth. In either case this amounts to observing the receiver output for one correlation interval. If the output exceeds the bias level, the signal is observed or detected (see pages 9-14 of RA-15061, *A Statistical Theory of Target Detection by Pulsed Radar*⁽²⁸⁾, hereafter referred to as No.1, for complete definitions of detection and bias level).

It will now be shown that the probability of detecting a given signal x is independent of the detector function, everything else being held constant and only one variate being taken from the density function of Eq.(8). The false alarm time has been defined as the time in which the probability is 1/2 that the noise alone will not exceed the bias level (Eq.(15), No.1), but it will be best here to keep things general and denote this probability as P_0 , rather than as 1/2. Eq.(15), No.1, then becomes

$$P = 1 - P_0^{N/n} \equiv \Gamma_N \quad (10)$$

where the subscript N denotes the number of variates and Γ is simply an abbreviation. From Eq.(8),

$$\Gamma_1 = \int_{y_b}^{F(\infty)} f(0, y) dy \quad (11)$$

where the symbol y_b is now used for the bias number. Then the probability of detection is

$$P_1 = \int_{y_b}^{F(\infty)} f(a, y) dy \quad (12)$$

but since $y = F(v)$, or $v = g(y)$, Eq.(11) may be written

$$\Gamma_1 = \int_{g(y_b)}^{\infty} v e^{-\frac{v^2}{2}} dv = e^{-\frac{g^2(y_b)}{2}} \quad (13)$$

and

$$g(y_b) = \sqrt{2 \log_e \frac{1}{\Gamma_1}} \quad (14)$$

Therefore Eq.(12) becomes

$$P_1 = \int_{\frac{1}{\sqrt{2 \log_e 1/\Gamma_1}}}^{\infty} v e^{-\frac{v^2+a^2}{2}} I_0(av) dv \quad (15)$$

which is independent of the detector function.

The integral of Eq.(15) must be evaluated by approximate methods. This function will appear in several places subsequently, and is defined as*

$$Q(\alpha, \beta) = \int_{\beta}^{\infty} v e^{-\frac{v^2+a^2}{2}} I_0(av) dv \quad (16)$$

Footnote on Q Functions

* It does not appear possible to express the Q function in terms of a finite number of known functions. The Q function is similar to Lommel's functions and in fact can be expressed as

$$Q(\alpha, \beta) = 1 - e^{-\frac{\alpha^2+\beta^2}{2}} \left[iU_1(-i\beta^2, i\alpha\beta) - U_2(-i\beta^2, i\alpha\beta) \right]$$

where U_1 and U_2 are Lommel's functions of the first kind. This identity may be proven using the definite integrals given in Watson⁽¹⁾, pages 540 and 541, especially Eq.5 of page 541. By successive integration by parts, the Q function may be expanded in infinite series giving

$$Q(\alpha, \beta) = e^{-\frac{\alpha^2+\beta^2}{2}} \sum_{r=0}^{\infty} \left(\frac{\alpha}{\beta} \right)^r I_r(\alpha\beta)$$

or

$$Q(\alpha, \beta) = 1 - e^{-\frac{\alpha^2+\beta^2}{2}} \sum_{r=1}^{\infty} \left(\frac{\beta}{\alpha} \right)^r I_r(\alpha\beta) \quad .$$

The similarity of the first of these expansions to the series for $U_1(w, z)$ given in Eq.(1), page 537 of Watson, is interesting. A simple expression for $Q(\alpha, \alpha)$ analogous to Eqs.(9) and (10), page 538 of Watson is

$$Q(\alpha, \alpha) = \frac{1}{2} \left[1 + e^{-\alpha^2} I_0(\alpha^2) \right] \quad (\text{Continued on next page.})$$

In terms of this notation, Eq.(15) may be written

$$P_1 = Q(a, \sqrt{2 \log_e 1/\Gamma_1}) . \quad (17)$$

This is the probability of detection only if $\tau_d = 1/f_r$ where τ_d is the time available for detection. In general the probability of detection is given by Eq.(18), No.1,

$$P_{N,\gamma} = 1 - (1 - P_N)^{\gamma'} \quad (18)$$

which follows from the definition of detection given on page 9, No.1. The double subscript notation $P_{N,\gamma}$ is used here in place of P' . If $N = \gamma$, $P_{N,N}$ is written simply as P_N .

$P_{1,\gamma}(R/R_0)$ can be calculated by means of Eqs.(18), (17), the tables of Q , and the simple relation

$$\frac{R}{R_0} = \frac{1}{x^{1/4}} = \frac{2^{1/4}}{\sqrt{a}} \quad (19)$$

(see Eqs.(10), (11) and (12), No.1).

Footnote on Q Functions (Cont'd)

which is useful in special cases. An asymptotic expansion for Q which is of value is given by Rice⁽¹⁸⁾, page 109:

$$Q(\alpha, \beta) = \frac{1}{2}[1 - \phi^{-1}(\beta - \alpha)] + \frac{1}{2\alpha\sqrt{2\pi}} e^{-\frac{(\beta - \alpha)^2}{2}} \left[1 - \frac{\beta - \alpha}{4\alpha} + \frac{1 + (\beta - \alpha)^2}{8\alpha^2} \dots \right]$$

where $\phi^{-1}(T)$ is given by the error function of Eq.(100). This expression is most useful in the region where $\alpha\beta \gg 1$ and $\alpha \gg |\beta - \alpha|$.

The Q function is a special case of the incomplete Toronto function described in the footnote on page 28. The relation is

$$Q(\alpha, \beta) = 1 - T_{\frac{\alpha}{\sqrt{2}}}\left(1, 0, \frac{\alpha}{\sqrt{2}}\right) .$$

The Q function is graphed in Figs.13 and 14.

A table is available in Ref.47 but the intervals are too large to be of general use. Project RAND is computing an extensive table of the Q function which will be published as a separate report.

A very good approximation for the quantity Γ_N , which appears in Eqs.(10) and (17), may be derived from Eq.(10) by writing

$$P_0^{1/n'} e^{1/n' \log_e P_0} \approx 1 - \frac{1}{n'} \log_e \frac{1}{P_0} \quad (20)$$

which is valid when $n' \gg 1$, a condition nearly always true in practice. Eq.(10) then becomes

$$\Gamma_N \approx \frac{N}{n} \log_e \frac{1}{P_0} \quad (21)$$

If $P_0 = 1/2$, as is assumed in all the curves in No.1,

$$\Gamma_1 = \frac{0.693}{n} \quad (22)$$

Eq.(17) may consequently be written

$$P_1 = Q \left[\sqrt{2} \left(\frac{R_0}{R} \right)^2, \sqrt{4.60 \log_{10} n + 0.730} \right] \quad (23)$$

As an example, let $R/R_0 = 0.595$, and $n = 10^4$. Then $P_1 = Q(4, 4.37)$, which has the value 0.410 from the table given in Ref.47. Note that this is a point on the graph of Fig.1, page 22, No.1.

GENERAL CASE — INTEGRATION OF N INDEPENDENT VARIATES

If the output of the receiver (or filter) can be observed for a length of time much greater than one correlation period, it is of advantage to integrate the output. The simplest concept of an integrator is a device which linearly adds the voltage output of N samples from the detector. The time elapsing between samplings must be at least one correlation period, in order that the samples may be considered to be independent. If the sum of N variates* of signal-plus-noise exceeds the bias level calculated from the probability density function for N variates of noise alone, then the signal is said to be detected.

* Readers with some statistical experience will recognize that here is a case of testing a statistical hypothesis. It is known that the n observations y_1, y_2, \dots, y_n come from a universe whose density function $f(y, a)$ depends on the unknown parameters a ; it is required to decide, on the basis of these observations, which of the two values a_1 or a_2 is a better estimate for a . If a_1 is the true value of a , let p_1 be the probability of making the mistake of choosing a_2 as the correct value; similarly, if a_2 is the true value, let p_2 be the probability of choosing a_1 . Suppose $p_1 = .05$. Then a statistical decision method can be devised for which $p_1 = .05$ and for which p_2 will be less than for any other method with the same p_1 . See, for example, Kendall, vol.2, pp.272-275(6).

The integrator may take the sum of the squares of the N variates, or, in general, the sum of N variates where each variate has been processed by some general function. As long as the same *weight* is applied to each variate, the integrator will be called linear. The function which the integrator applies to each variate will be called the *law* of the integrator. Any nonlinear integrator will be inferior in operation to a linear integrator with the same law and would ordinarily never be used intentionally in practice. Cathode ray tubes are nonlinear, however, and thus fall short of other types of linear integrations.

The *law* of the integrator acts in exactly the same way as the law of the detector. Thus, if the detector output is $y = (F(v))$ as given by Eq.(5), the integrator output is

$$Y = \sum_1^N \phi(y) = \sum_1^N \phi[F(v)]. \quad (24)$$

It is obvious, as far as the theoretical problem is concerned, that the only function of importance is

$$\psi(v) = \phi[F(v)]. \quad (25)$$

There will be an infinite number of combinations of ϕ and F functions which will produce the same function ψ and hence the same theoretical results. In all the work that follows, the output of the combination of integrator and detector for one independent variate will be called $y = \psi(v)$, and the sum of N variates will be

$$Y = \sum_1^N y. \quad (26)$$

The symbolic solution for the case of N variates corresponding to Eq.(15) for one variate is not too difficult to obtain. The starting point is Eq.(8) for the probability density function for one variate. The characteristic function for this distribution is

$$C_1 = \int_{-\infty}^{\infty} f(a, y) e^{i\omega y} dy. \quad (27)$$

The characteristic function for the probability density function for the sum of N independent variates is then simply

$$C_N = (C_1)^N \quad (28)$$

and

$$dP_N = dY \int_{-\infty}^{\infty} C_N(a, \omega) e^{-i\omega y} \frac{d\omega}{2\pi} \quad (29)$$

or

$$dP_N = G(a, N, Y) dY . \quad (30)$$

Corresponding to Eq.(11) is

$$\Gamma_N = \int_{Y_b}^{\infty} G(0, N, Y) dY \quad (31)$$

and to Eq.(12),

$$P_N = \int_{Y_b}^{\infty} G(a, N, Y) dY . \quad (32)$$

If Y_b is eliminated from Eqs.(31) and (32), there results a solution for P_N as a function of Γ_N , N , and a , which is the desired result.

It is found in most cases that the integrations required in Eqs.(27) to (32) are not possible in terms of known functions.

THE SQUARE LAW DETECTOR WITH N VARIATES

It seems, by a process of trial and error, that the best possible function for $\psi(v)$ in Eq.(25) to produce integrable functions in Eqs.(27) to (32) is

$$\psi(v) = Av^2 = y . \quad (33)$$

Though this represents a square law for the combined detector and integrator law, it is usual to think of it as representing a square law detector coupled with a linear law integrator.

In Eq.(33), the only effect of the constant A is to multiply the bias level Y_b in Eqs.(31) and (32) by A . The value of P_N in Eq.(32) is independent of A . It is convenient to let $A = 1/2$, or $y = v^2/2$. By direct substitution from Eqs.(6) and (27),

$$C_1 = \int_0^{\infty} e^{-y-x} I_0(2\sqrt{xy}) e^{py} dy \quad (34)$$

where $x = a^2/2$ and $p = i\omega$.

This integral may be obtained from pair 655.1 of Campbell and Foster⁽⁷⁾. In all pairs taken from Campbell and Foster it is necessary to replace p by $-p$, since they use e^{pB} for the first integration. As long as the same notation is used in both directions, the order of signs is immaterial. In order to avoid confusion, the minus sign will be used in the exponent in the first transformation and the plus sign in the second transformation. Thus all of the characteristic functions which appear hereafter are really $C(-p)$ rather than $C(p)$. In this way there is direct agreement with the Campbell and Foster tables as well as with tables of the Laplace transform. Equation (34) becomes

$$C_1 = \frac{1}{p+1} e^{-x} e^{\frac{x}{p+1}} \quad (35)$$

The characteristic function for the sum of N variates is then simply

$$C_N = \frac{e^{-Nx}}{(p+1)^N} e^{\frac{Nx}{p+1}} \quad (36)$$

By means of pair 650.0, Campbell and Foster, the probability density function is

$$\begin{aligned} dP_N &= \left(\frac{Y}{Nx}\right)^{\frac{N-1}{2}} e^{-Y-Nx} I_{N-1}(2\sqrt{NxY}) dY & Y > 0 \\ &= 0 & Y < 0 \end{aligned} \quad (37)$$

Graphs of this function are shown in Figs.1-7. The density function for noise alone ($x = 0$) is found most easily from pair 431, Campbell and Foster, to be

$$dP_N = \frac{Y^{N-1} e^{-Y}}{(N-1)!} dY \quad (38)$$

BIAS LEVEL FOR SQUARE LAW CASE

The bias level, Y_b , is by Eq.(31)

$$\Gamma_N = \int_{Y_b}^{\infty} \frac{Y^{N-1} e^{-Y}}{(N-1)!} dY \quad . \quad (39)$$

The incomplete gamma function, as defined by Pearson⁽⁸⁾, is

$$I(u, p) = \int_0^{u\sqrt{p+1}} \frac{e^{-v} v^p dv}{p!} \quad . \quad (40)$$

In terms of this function, Eq.(39) becomes

$$\Gamma_N = 1 - I\left(\frac{Y_b}{\sqrt{N}}, N-1\right) \quad . \quad (41)$$

The tables of $I(\mu, p)$ extend to $p = 50$, and values of the function are given to seven places. Thus, for $N < 50$, and $n' < 10^6$, the bias level Y_b may be obtained directly from Pearson's tables. Other methods must be evolved for $N > 50$ or $n' > 10^6$. The normal approximation to Eq.(39) is unsatisfactory for N less than several thousand because of the fact that the integral is over a region which is far out on the tail of the curve. This can be seen from the Gram-Charlier series which will be taken up presently.

The integral of Eq.(39) may be evaluated directly by successive integration by parts to give

$$\Gamma_N = \frac{Y_b^{N-1} e^{-Y_b}}{(N-1)!} = \left[1 + \frac{N-1}{Y_b} + \frac{(N-1)(N-2)}{Y_b^2} \text{-----} \right] \quad . \quad (42)$$

In the regions of interest $Y_b > N \gg 1$. The series in the brackets may be approximated by

$$1 + \frac{N-1}{Y_b} + \text{-----} \approx \frac{1}{1 - \frac{N-1}{Y_b}} \quad (43)$$

in this region so that Eq.(42) becomes

$$\Gamma_N \approx \frac{Y_b^{N-1} e^{-Y_b}}{(N-1)! \left(1 - \frac{N-1}{Y_b}\right)} = \frac{NY_b^N e^{-Y_b}}{N! (Y_b - N + 1)} \quad (44)$$

By the use of Sterling's approximation for $N!$, Eq.(44) reduces to

$$\Gamma_N \approx \sqrt{\frac{N}{2\pi}} \frac{\exp \left[-Y_b + N \left(1 + \log_e \frac{Y_b}{N} \right) \right]}{(Y_b - N + 1)} \quad (45)$$

Though the expression looks more cumbersome in this form, it is actually much simpler to use in calculations than is Eq.(44). Substituting for Γ_N from Eq.(21) gives the expression

$$\begin{aligned} \log_{10} n &= 0.24 + \frac{1}{2} \log_{10} N + \log_{10} (Y_b - N + 1) \\ &+ 0.434 (Y_b - N) - N \log_{10} \frac{Y_b}{N} \quad (46) \end{aligned}$$

Graphs of this function are shown in Figs.8 and 9. For $N = 1$, the exact expression for Y_b from Eq.(39) is

$$Y_b = 2.3 \log_{10} n + 0.37 \quad (47)$$

whereas Eq.(46) for $N = 1$ reduces to

$$Y_b = 2.3 \log_{10} n + 0.45 \quad (48)$$

The difference is seen to be practically negligible.

CUMULATIVE DISTRIBUTION FOR N VARIATES OF
SIGNAL PLUS NOISE - SQUARE LAW DETECTOR

Knowing the required bias level for a given false alarm interval, it is now necessary to integrate the density function of Eq.(37) from this value to infinity to give the probability of detection for a signal of strength x , thus

$$P_N = \int_{Y_h}^{\infty} \left(\frac{Y}{Nx}\right)^{\frac{N-1}{2}} e^{-Y-Nx} I_{N-1}(2\sqrt{NxY}) dY . \quad (49)$$

This integral is not soluble directly in terms of well-known functions. The order of the Bessel function* can be reduced in steps of 1 by successive integrations by parts, so that the last remaining integral is of the type given by the Q function

* The following are some of the useful identities concerning the modified Bessel functions of the first kind:

$$I_n(z) = (-i)^n J_n(iz) = \sum_{r=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{n+2r}}{r!(n+r)!} = \frac{\left(\frac{z}{2}\right)^n}{N!} \left[1 + \frac{z^2}{2(2n+2)} + \frac{z^4}{2 \cdot 4(2n+2)(2n+4)} \dots \right]$$

$$I_0(z) = 1 + \frac{z^2}{4} + \frac{z^4}{64} \dots$$

Asymptotic expansion:

$$I_n(x) \approx \frac{e^x}{\sqrt{2\pi x}} \left[1 + \frac{1-4n^2}{1!(8x)} + \frac{(1-4n^2)(9-4n^2)}{2!(8x^2)} \dots \right]$$

$$I_n(x) = \frac{1}{\sqrt{\pi} \Gamma\left(n+\frac{1}{2}\right)} \left(\frac{x}{2}\right)^n \int_0^{\pi} e^{\pm x \cos \psi} (\sin \psi)^{2n} d\psi$$

$$z I_n'(z) = n I_n(z) + z I_{n+1}(z) = -n I_n(z) + z I_{n-1}(z)$$

$$I_0'(z) = I_1(z)$$

$$\int z^n I_{n-1}(z) dz = z^n I_n(z)$$

$$\int z I_0(z) dz = z I_1(z)$$

$$\frac{2n}{z} I_n(z) = I_{n-1}(z) - I_{n+1}(z)$$

$$\int e^x I_0(x) dx = x e^x [I_0(x) - I_1(x)]$$

$$\int e^{-x} I_0(x) dx = x e^{-x} [I_0(x) + I_1(x)]$$

$$\int e^x I_1(x) dx = e^x [(1-x) I_0(x) + x I_1(x)]$$

$$\int e^{-x} I_1(x) dx = e^{-x} [(1+x) I_0(x) + x I_1(x)]$$

Relations between the I_n functions and the hypergeometric functions will be found in the footnote on p. 21.

of Eq.(16). An easier way to arrive at the same result is by the use of the characteristic function. To get the cumulative distribution from $-\infty$ to Y of any density function, it is only necessary to find the anticharacteristic function of C/p , where C is the characteristic function of the given density function (see pair 210, Campbell and Foster). Thus from Eq.(36),

$$P_N = 1 - \int_{-\infty}^{\infty} \frac{e^{-Nx}}{p(p+1)^N} e^{\frac{Nx}{p+1}} e^{pY_b} df^* . \quad (50)$$

The term $1/p(p+1)^N$ may be expanded in a series

$$\frac{1}{p(p+1)^N} = \frac{1}{p(p+1)} - \frac{1}{(p+1)^2} + \frac{1}{(p+1)^3} - \dots - \frac{1}{(p+1)^N} . \quad (51)$$

The mate of the first term of the series, by pairs 210, and 655.1, Campbell and Foster, is

$$e^{-Nx} \int_0^{Y_b} e^{-y} I_0(2\sqrt{Nxy}) dy . \quad (52)$$

The first two terms of P_N are thus

$$\begin{aligned} & 1 - \int_0^{Y_b} e^{-y-Nx} I_0(2\sqrt{Nxy}) dy \quad (53) \\ &= \int_{\sqrt{2Y_b}}^{\infty} v e^{-\frac{v^2+2Nx}{2}} I_0(v\sqrt{2Nx}) dv \\ &= Q(\sqrt{2Nx}, \sqrt{2Y_b}) \end{aligned}$$

using the definition of Q from Eq.(16). All the succeeding terms may be obtained by using pair 650.0 Campbell and Foster.

$$\text{Mate of } \frac{e^{\frac{Nx}{p+1}}}{(p+1)^r} \text{ is } \left(\frac{y}{Nx}\right)^{\frac{r-1}{2}} e^{-y} I_{r-1}(2\sqrt{Nxy}) . \quad (54)$$

* As in Campbell and Foster, f is here used in place of $\omega/2\pi$ or $p/2\pi i$.

From Eqs. (53) and (54),

$$P_N = Q(\sqrt{2Nx}, \sqrt{2Y_b}) + e^{-Y_b - Nx} \sum_{r=2}^{r=N} \left(\frac{Y_b}{Nx}\right)^{\frac{r-1}{2}} I_{r-1}(2\sqrt{NxY_b}) \quad (55)$$

This form of solution for P_N is practical for numerical calculation only where N is less than about 10.

The characteristic function in Eq. (50) can be expanded in another manner using

$$\frac{1}{p(p+1)^N} = \sum_{r=N+1}^{r=\infty} \frac{1}{(p+1)^r} \quad (56)$$

This leads to an expression for P_N complementary to that of Eq. (55) of the form

$$P_N = 1 - e^{-Y_b - Nx} \sum_{r=N+1}^{r=\infty} \left(\frac{Y_b}{Nx}\right)^{\frac{r-1}{2}} I_{r-1}(2\sqrt{NxY_b}) \quad (57)$$

By equating (55) and (57), one obtains one of the known expansions for Q given in the footnote on page 5. Equations (55) and (57) may also be obtained directly from Eq. (49) by repeated integration by parts. Equation (57) may also be obtained directly from Eq. (55) by means of the identity

$$e^{\frac{x}{2}\left(t + \frac{1}{t}\right)} = \sum_{n=-\infty}^{n=\infty} t^n I_n(x) \quad (58)$$

given in McRobert⁽²⁾, page 32, and one of the known series for Q .

For the special case $Y_b = Nx$, the function Q of Eq. (55) is simply

$$Q = \frac{1}{2} \left[1 + e^{-2Y_b} I_0(2Y_b) \right] \quad (59)$$

(see footnote, page 5), and Eq. (55) becomes

$$P_N = \frac{1}{2} + e^{-2Y_b} \left[\frac{1}{2} I_0(2Y_b) + I_1(2Y_b) + I_2(2Y_b) + \dots + I_{N-1}(2Y_b) \right] \quad (60)$$

This formula is useful for checking special points for values of N around 10 or below.

None of the methods developed above are suitable for calculating P_N for large values of N .

In the next section the general method of Gram-Charlier series is developed, which will be useful in a number of succeeding problems concerning distribution functions over large ranges of variation of N .

EXPANSION OF FUNCTIONS IN GRAM-CHARLIER SERIES

The function $\phi(y)$ is defined by

$$\phi(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \quad (61)$$

The Hermite polynomials may be defined by the relation

$$\phi^i(y) = \frac{(-1)^i}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} H_i(y) \quad (62)$$

where the superscript i stands for the i^{th} derivative with respect to y . The ϕ functions and the Hermite polynomials are biorthogonal, that is

$$\int_{-\infty}^{\infty} H_i(y) \phi^j(y) dy = \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \quad (63)$$

Therefore it is possible to expand any reasonable function in a series of the form⁽⁵⁾

$$f(y) = \sum_{i=0}^{i=\infty} a_i \phi^i(y) \quad (64)$$

The coefficients a_i may be evaluated in a manner analogous to the Fourier series methods by multiplying both sides of Eq. (64) by $H_i(y)$ and integrating from $-\infty$ to ∞ . All terms drop out but one, giving

$$a_i = \frac{(-1)^i}{i!} \int_{-\infty}^{\infty} H_i(y) f(y) dy \quad (65)$$

It is usual to make the substitution $t = y - \bar{y}/\sigma$ before making the expansion, thus causing the second and third terms of the series to vanish. The notation is

$\bar{y} = \nu_1$ = the average value of y , or the first moment

$\sigma^2 = \nu_2 - \nu_1^2$ = the variance

ν_n = the n^{th} moment of the distribution.

$$\nu_n = \int_{-\infty}^{\infty} y^n f(y) dy \quad (66)$$

Equation (64) is replaced by

$$f(y) = g(t) = \sum_{i=0}^{i=\infty} c_i \phi^i(t) \quad (67)$$

and Eq.(65) by

$$c_i = \frac{(-1)^i}{i!} \int_{-\infty}^{\infty} H_i(t) g(t) dt = \frac{(-1)^i}{i!} \int_{-\infty}^{\infty} H_i\left(\frac{y-\bar{y}}{\sigma}\right) f(y) \frac{dy}{\sigma} \quad (68)$$

It follows at once from Eq.(68) that $c_0 = 1/\sigma$, $c_1 = c_2 = 0$. The moments about the mean, or the central moments, are defined by

$$\mu_i = \int_{-\infty}^{\infty} (y-\bar{y})^i f(y) dy \quad (69)$$

and the standard moments about the mean by

$$a_i = \frac{\mu_i}{\sigma^i} \quad (70)$$

The coefficients c_i in Eq.(68) may be easily written in terms of the α 's. The first few are

$$c_3 = -\frac{1}{3!} \alpha_3 \quad (71a)$$

$$c_4 = \frac{1}{4!} (\alpha_4 - 3) \quad (71b)$$

$$c_5 = -\frac{1}{5!} (\alpha_5 - 10\alpha_3) \quad (71c)$$

$$c_6 = \frac{1}{6!} (\alpha_6 - 15\alpha_4 + 30) \quad (71d)$$

$$c_7 = -\frac{1}{7!} (\alpha_7 - 21\alpha_5 + 105\alpha_3) \quad (71e)$$

$$c_8 = \frac{1}{8!} (\alpha_8 - 28\alpha_6 + 210\alpha_4 - 315) \quad (71f)$$

$$c_9 = -\frac{1}{9!} (\alpha_9 - 36\alpha_7 + 378\alpha_5 - 1260\alpha_3) \quad (71g)$$

Formulae for the μ 's in terms of the ν 's can be obtained directly from Eq.(69), giving

$$\mu_2 = \nu_2 - \nu_1^2 \quad (72a)$$

$$\mu_3 = \nu_3 - 3\nu_2\nu_1 + 2\nu_1^3 \quad (72b)$$

$$\mu_4 = \nu_4 - 4\nu_3\nu_1 + 6\nu_2\nu_1^2 - 3\nu_1^4 \quad (72c)$$

$$\mu_5 = \nu_5 - 5\nu_4\nu_1 + 10\nu_3\nu_1^2 - 10\nu_2\nu_1^3 + 4\nu_1^5 \quad (72d)$$

$$\mu_6 = \nu_6 - 6\nu_5\nu_1 + 15\nu_4\nu_1^2 - 20\nu_3\nu_1^3 + 15\nu_2\nu_1^4 - 5\nu_1^6 \quad (72e)$$

Continuations of this series are obvious.

The process of obtaining the Gram-Charlier expansion is now evident:

1. Find the moments of the distribution.
2. Obtain the central moments from Eq.(72).
3. Obtain the standard central moments from Eq.(70).
4. Obtain the coefficients from Eq.(71).
5. Write the series for $f(y)$ from Eq.(67).

It turns out that the best grouping for the terms of the series of Eq.(67) is different from the natural sequence⁽⁶⁾. Such a regrouped series is termed an "Edgeworth series" and is actually used in this work. The grouping used by Edgeworth is

$$0 \quad (73a)$$

$$0, 3 \quad (73b)$$

$$0, 3, 4, 6 \quad (73c)$$

$$0, 3, 4, 6, 5, 7, 9 \quad (73d)$$

This means that if the 0 and 3 terms are used as the first approximation, the addition of terms 4 and 6 gives the next order approximation, and so forth.

MOMENTS OF SIGNAL PLUS NOISE, SQUARE LAW DETECTOR

The moments of a distribution may be obtained by using the characteristic function as a moment generating function*. Thus

$$\nu_i = (-1)^i \left(\frac{d^i C}{dp^i} \right)_{p=0} \quad (74)$$

* Kendall, p.54⁽⁶⁾.

In the case of the distribution function for the sum of N variates of signal plus noise with a square law detector, the characteristic function is given by Eq.(36), and the moments are

$$\nu_i = (-1)^i \left[\frac{d^i}{dp^i} \frac{e^{-Nx} e^{\frac{Nx}{p+1}}}{(p+1)^N} \right]_{p=0} \quad (75)$$

Though the first few moments may be obtained by direct differentiation, it is better in this case to expand in a McLaurin's series and obtain the coefficient of $p^i/i!$. Thus

$$\frac{e^{\frac{Nx}{p+1}}}{(p+1)^N} = \frac{1}{(p+1)^N} + \frac{Nx}{(p+1)^{N+1}} + \frac{(Nx)^2}{(p+1)^{N+2} \cdot 2!} \quad (76)$$

The coefficient of $p^i/i!$ is, by direct expansion of each term in Eq.(76),

$$(-1)^i \frac{(N+i-1)!}{(N-1)!} \left[1 + \frac{(N+i)}{N} Nx + \frac{(N+i)(N+i+1)}{N(N+1)} \frac{(Nx)^2}{2!} + \dots \right] \quad (77)$$

$$= (-1)^i \frac{(N+i-1)!}{(N-1)!} {}_1F_1(N+i, N, Nx) \quad (78)$$

where ${}_1F_1$ is the confluent hypergeometric function.* Thus the moments are

* The following are some of the useful relations concerning the confluent hypergeometric function:

$${}_1F_1(a, c, z) = \frac{\Gamma(c)}{\Gamma(a)} \sum_{r=0}^{\infty} \frac{\Gamma(a+r)}{r! \Gamma(c+r)} z^r = 1 + \frac{a}{c} \frac{z}{1!} + \frac{a(a+1)}{c(c+1)} \frac{z^2}{2!} + \dots = {}_2F_1(a, b, c, \frac{z}{b})$$

Lim $b \rightarrow \infty$

Asymptotic expansion:

$${}_1F_1(a, c, z) \approx \frac{\Gamma(c) e^z}{\Gamma(a) z^{c-a}} \left[1 + \frac{(1-a)(c-a)}{z 1!} + \dots \right]$$

Kummer's first transformation:

$${}_1F_1(a, c, z) = e^z {}_1F_1(c-a, c, -z)$$

Kummer's second transformation:

$${}_1F_1(a, 2a, 2z) = e^z {}_0F_1\left(a + \frac{1}{2}, \frac{z^2}{4}\right)$$

Recursion relations:

$$a {}_1F_1(a+1, c, z) + (a-c) {}_1F_1(a-1, c, z) = (z+2a-c) {}_1F_1(a, c, z)$$

$$ac {}_1F_1(a+1, c, z) + (c-a)z {}_1F_1(a, c+1, z) = c(a+z) {}_1F_1(a, c, z)$$

(Continued on next page.)

$$\nu_i = \frac{(N+i-1)!}{(N-1)!} e^{-Nx} {}_1F_1(N+i, N, Nx) \quad (79)$$

or

$$\nu_i = \frac{(N+i-1)!}{(N-1)!} {}_1F_1(-i, N, -Nx) \quad (80)$$

Hypergeometric function (Cont'd)

$$a {}_1F_1(a+1, c, z) + (1-c) {}_1F_1(a, c-1, z) = (a+1-c) {}_1F_1(a, c, z)$$

$$c {}_1F_1(a-1, c, z) + z {}_1F_1(a, c+1, z) = c {}_1F_1(a, c, z)$$

$$(a-c) {}_1F_1(a-1, c, z) + (c-1) {}_1F_1(a, c-1, z) = (z+a-1) {}_1F_1(a, c, z)$$

$$(c-a) z {}_1F_1(a, c+1, z) + c(c-1) {}_1F_1(a, c-1, z) = c(z+c-1) {}_1F_1(a, c, z)$$

$$\frac{d}{dz} {}_1F_1(a, c, z) = \frac{a}{c} {}_1F_1(a+1, c+1, z)$$

Relations between hypergeometric functions and other functions:

$${}_1F_1(a, a, z) = e^z$$

$${}_1F_1(a, a+1, -z) = az^{-a} \int_0^z e^{-t} t^{a-1} dt = z^{-a} \Gamma(a+1) I\left(\frac{z}{a}, a-1\right)$$

using Pearson's notation for the incomplete gamma function.

$${}_1F_1(1, a+1, z) = e^{-z} z^a \Gamma(a+1) I\left(\frac{z}{a}, a-1\right)$$

$${}_1F_1\left(\frac{1}{2}, \frac{3}{2}, -z^2\right) = \frac{1}{z} \int_0^z e^{-t^2} dt = \frac{\sqrt{\pi}}{2z} \operatorname{erf} z$$

$${}_1F_1(-n, 1, z) = L_n(z)$$

(original Laguerre polynomial)

$${}_1F_1(-n, a+1, z) = \frac{n! \Gamma(a+1)}{\Gamma(a+1+n)} L_n^a(z)$$

(generalized Laguerre polynomial)

$${}_1F_1\left(\nu+\frac{1}{2}, 2\nu+1, -z\right) = \frac{2^{2\nu} \Gamma(\nu+1) e^{-\frac{z}{2}}}{(-z)^\nu} I_\nu\left(\frac{z}{2}\right)$$

$${}_1F_1\left(\frac{1}{2}, 1, -z\right) = e^{-\frac{z}{2}} I_0\left(\frac{z}{2}\right)$$

$${}_1F_1\left(\frac{1}{2}, 2, -z\right) = e^{-\frac{z}{2}} \left[I_0\left(\frac{z}{2}\right) + I_1\left(\frac{z}{2}\right) \right]$$

$${}_1F_1\left(-\frac{1}{2}, 1, -z\right) = e^{-\frac{z}{2}} \left[(1+z) I_0\left(\frac{z}{2}\right) + z I_1\left(\frac{z}{2}\right) \right]$$

$${}_1F_1\left(\frac{3}{2}, 1, -z\right) = e^{-\frac{z}{2}} \left[(1-z) I_0\left(\frac{z}{2}\right) + z I_1\left(\frac{z}{2}\right) \right]$$

$${}_1F_1\left(\frac{3}{2}, 2, -z\right) = e^{-\frac{z}{2}} \left[I_0\left(\frac{z}{2}\right) - I_1\left(\frac{z}{2}\right) \right]$$

Eq.(80) being obtained from Eq.(79) by Kummer's first transformation. The first four moments are

$$\nu_1 = N(1+x) \quad (81a)$$

$$\nu_2 = (Nx)^2 + 2Nx(N+1) + N(N+1) \quad (81b)$$

$$\nu_3 = (Nx)^3 + 3(Nx)^2(N+2) + 3Nx(N+1)(N+2) + N(N+1)(N+2) \quad (81c)$$

$$\begin{aligned} \nu_4 = (Nx)^4 + 4(Nx)^3(N+3) + 6(Nx)^2(N+2)(N+3) \\ + 4Nx(N+1)(N+2)(N+3) + N(N+1)(N+2)(N+3) . \end{aligned} \quad (81d)$$

The generalized Laguerre polynomial $L_n^{(\alpha)}(z)$ is defined by

$$L_n^{(\alpha)}(z) = \frac{\Gamma(\alpha+1+n)}{n!\Gamma(\alpha+1)} {}_1F_1(-n, \alpha+1, z) . \quad (82)$$

Comparing (80) and (82), it is seen that the moments expressed in terms of the Laguerre polynomials are

$$\nu_i = i! L_i^{(N-1)}(-Nx) . \quad (83)$$

Another generating function for these polynomials is available through the relation

$$L_n^{(\alpha)}(z) = \frac{e^z z^{-\alpha}}{n!} \frac{d^n}{dz^n} (e^{-z} z^{n+\alpha}) . \quad (84)$$

The moments about the mean may be expressed in terms of the moments about the origin by means of Eqs.(72a-e), resulting in:

$$\mu_0 = 1 \quad (85a)$$

$$\mu_1 = 0 \quad (85b)$$

$$\mu_2 = 2Nx + N = N(2x+1) = \sigma^2 \quad (85c)$$

$$\mu_3 = 6Nx + 2N = 2N(3x+1) \quad (85d)$$

$$\mu_4 = 12(Nx)^2 + 12Nx(N+2) + 3N(N+2) \quad (85e)$$

$$\mu_5 = 120(Nx)^2 + 20Nx(5N+6) + 4N(5N+6) \quad (85f)$$

A generating function for the central moments may be obtained by multiplying the generating function of Eq.(75) by e^{pv} giving (see pair 207, Campbell and Foster)

$$\mu_i = (-1)^i e^{-Nx} \left[\frac{d^i}{dp^i} \frac{e^{\frac{Nx}{p+1} + p(Nx+N)}}{(p+1)^N} \right]_{p=0} \quad (86)$$

The moments of Eqs.(85a-f) are most easily obtained by logarithmic differentiation in Eq.(86).

The standard moments about the mean are obtained from Eq.(70), and are

$$a_0 = 1 \quad (87a)$$

$$a_1 = 0 \quad (87b)$$

$$a_2 = 1 \quad (87c)$$

$$a_3 = \frac{2(3x+1)}{N^{1/2}(2x+1)^{\frac{3}{2}}} \quad (87d)$$

$$a_4 = 3 + \frac{6(4x+1)}{N(2x+1)^2} \quad (87e)$$

There is an approximate method of computing the significant part of α_6 which is based on the fact that c_6 of Eq.(71d) is always nearly equal to $c_3^2/2$ (see page 259, Fry⁽⁵⁾). Thus

$$\alpha_6 \approx 15\alpha_4 + 10\alpha_3^2 - 30 \quad (88)$$

or

$$\alpha_6 \approx 15 + \frac{10(108x^2 + 78x + 13)}{N(2x+1)^3} \quad (87f)$$

For noise alone, the moments are given by

$$\nu_i = \frac{(N+i-1)!}{(N-1)!} \quad (89)$$

and the central moments by

$$\mu_i = \frac{(N+i-1)!}{(N-1)!} {}_1F_1(-i, 1-i-N, -N) \quad (90)$$

Equation (90) was obtained from Eq.(86) by putting $x=0$ and expanding in a series.

THE GRAM-CHARLIER SERIES FOR THE SQUARE LAW CASE

The coefficients of the series may be obtained by use of Eqs.(71a-d) since the standard central moments are now known (Eqs.87a-f). They are:

$$c_0 = 1 \quad (91a)$$

$$c_1 = c_2 = 0 \quad (91b)$$

$$c_3 = - \frac{3x+1}{3N^{1/2}(2x+1)^{3/2}} \quad (91c)$$

$$c_4 = \frac{4x+1}{4N(2x+1)^2} \quad (91d)$$

$$c_6 = \frac{(3x+1)^2}{18N(2x+1)^3} \quad (91e)$$

From Eq.(67) the required series is

$$dP = \frac{dy}{\sigma} \left[c_0 \phi^0(t) + c_3 \phi^3(t) + c_4 \phi^4(t) + c_6 \phi^6(t) \dots \right] \quad (92)$$

where

$$t = \frac{y - \nu_1}{\sigma} \quad , \quad (93)$$

$$\nu_1 = N(1+x) \quad (94)$$

$$\sigma = \sqrt{N(1+2x)}$$

and the c 's are given by Eqs.(91a-e).

Note that the grouping of terms is according to the Edgeworth scheme given in Eq.(73). Note further that as N tends to infinity, all the coefficients go to zero except c_0 . Thus

$$dP = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-\nu_1)^2}{2\sigma^2}} dy \quad (95)$$

as $N \rightarrow \infty$. In terms of N and x

$$dP = \frac{1}{\sqrt{2\pi N(1+2x)}} e^{-\frac{[y-N(1+x)]^2}{2N(1+2x)}} dy \quad (96)$$

Eq.(95) is precisely a statement of the central limit theorem, and the derivation given is essentially a loose proof of the theorem.

The cumulative distribution is easily obtained from Eq.(92) by means of the simple relation

$$\int \phi^i(t) dt = \phi^{i-1}(t) \quad (97)$$

giving

$$P_N = \int_{Y_b}^{\infty} f(y) dy = \frac{1}{2} [1 - \phi^1(T)] - c_3 \phi^2(T) - c_4 \phi^3(T) - c_6 \phi^5(T) \text{ ---- (98)}$$

where

$$T = \frac{Y_b - \nu_1}{\sigma} \quad (99)$$

and

$$\phi^1(T) = \frac{1}{\sqrt{2\pi}} \int_{-T}^T e^{-\frac{a^2}{2}} da \quad (100)$$

The function $\phi^{-1}(T)$ is tabulated in the W.P.A. tables⁽⁹⁾. This differs from the definition given for $\phi^{-1}(y)$ in Fry, page 456 but is used here because of the W.P.A. tables.

The series of Eq.(98) was used to calculate all the curves of Figs.1-50, No.1, with the exception of the cases where $N = 1$. In most cases the first two terms of the series are sufficient, though in some regions of small P four terms are needed.

SAMPLE CALCULATION

Assume $N = 10$, $n = 10^6$

From Fig.8, or Eq.41, $Y_b = 30.0$

Let $\frac{R}{R_0} = 1.0$ so that $x = 1.0$.

From Eq. (94), $\nu_1 = N(1+x) = 20.0$
 $\sigma = \sqrt{N(1+2x)} = 5.48$

From Eq. (99), $T = \frac{Y_1 - \nu_1}{\sigma} = \frac{30.0 - 20.0}{5.58} = 1.830$

$$\phi^{-1}(T) = \phi^{-1}(1.828) = 0.9325$$

$$\frac{1}{2}(1 - \phi^{-1}(T)) = 0.0338$$

From Eq. (91c), $c_3 = -\frac{3x+1}{3N^{1/2}(2x+1)^{3/2}} = -0.081$

From Eq. (91d), $c_4 = \frac{4x+1}{4N(2x+1)^2} = 0.0139$

From Eq. (91e), $c_6 = \frac{(3x+1)^2}{18N(2x+1)^3} = 0.0033$

$$\phi^2(1.828) = 0.174$$

(See p.64 for references on tables of the derivatives of the error function.)

$$\phi^3(1.828) = -0.470$$

$$\phi^5(1.828) = 0.990$$

$$c_3 \phi^2(T) = -0.0141$$

$$c_4 \phi^3(T) = -0.0007$$

$$c_6 \phi^5(T) = +0.0032$$

$$P = 0.0338 + 0.0141 + 0.0007 - 0.0032 = 0.0452$$

This point, $P = 0.045$, $R/R_0 = 1$, may be found on Fig.20, No.1.

INTEGRATION LOSS, SQUARE LAW DETECTOR

It is of interest to express the effect of noncoherent integration as a loss with respect to coherent integration⁽⁴³⁾. This may be done by defining the integration loss as

$$L_i = 10 \log_{10} \frac{Nx_2}{x_1} \quad (100a)$$

where

N = number of pulses integrated

x_1 = required value of signal-to-noise ratio to produce given probability of detection for $N = 1$.

x_2 = required value of signal-to-noise ratio to produce the same probability of detection for $N = N$.

Thus L_i is a function of P and n . However, it turns out that the dependence on P and n is very small.

In the case of coherent integration, x_2 is always equal to x_1/N , so that $L_i = 0$. With noncoherent integration, x_2 is always greater than x_1/N , so that noncoherent integration is never as efficient as coherent integration. The results of calculations are given in Figs.10 and 11. One observes that the dependence of L_i on P and n is quite small. Thus by means of the graph of Fig.12, which gives x as a function of P and n for $N = 1$, and any one of the curves of Fig.10, it is possible to obtain a fairly accurate value of x for any P , n and N .

GENERAL CURVES OF THE CUMULATIVE DISTRIBUTION FUNCTION

The integral of Eq.(49) is a function found in other applications than the one discussed in this paper. It is desirable to have graphs of this function available in general form rather than the specialized form of Figs.1-50, No.1. The integral is a special case of the incomplete Toronto function* described by Heatley⁽⁴⁹⁾ and Fisher⁽¹⁷⁾, which is defined as

$$T_B(m, n, r) = 2r^{n-m+1}e^{-r^2} \int_0^B t^{m-n}e^{-t^2} I_n(2rt) dt \quad (100b)$$

Using this notation, Eq.(49) for the cumulative distribution function becomes

$$P_N = 1 - T_{\sqrt{V_0}}(2N-1, N-1, \sqrt{Nx}) \quad (100c)$$

* In normal correlation theory, the quantity

$$df = \left(\frac{B}{i\beta}\right)^{\frac{n-2}{2}} e^{-\frac{1}{2}(B^2-\beta^2)} J_{\frac{1}{2}(n-2)}(i\beta B) d\left(\frac{1}{2}B^2\right)$$

is given by Fisher⁽¹⁷⁾ as the limiting form, for large samples, of the frequency element of the quantity $B^2 = n_2 R^2$ where R^2 is the sample estimate of the multiple correlation coefficient of a random variable y with other variables x_1, x_2, \dots, x_{n_1} , n_2 is the size of the sample, and $\beta_2^2 = n_2 \rho^2$ where ρ is the population multiple correlation coefficient. The cumulative distribution is

$$f = T_{\frac{B}{\sqrt{2}}}\left(n_1-1, \frac{1}{2}n_1-1, \frac{\beta}{\sqrt{2}}\right)$$

and can be obtained from the curves in Figs.13 to 32.

The function plotted in Figs. 13 to 32 is

$$T_{\sqrt{v}}(2N-1, N-1, \sqrt{q}) \quad (100d)$$

and P_N may be found easily from these curves for any values of Y_b , N and x .

THE LINEAR DETECTOR - N VARIATES

The linear detector is usually more difficult to deal with than is the square law detector. The distribution function for one variate of signal-plus-noise is

$$dP = v e^{-\frac{v^2+a^2}{2}} I_0(av) dv . \quad (101)$$

In attempting to find the distribution for the sum of N variates by the method of characteristic functions, the immediate trouble is that the characteristic function of Eq. (101) does not seem to be obtainable in closed form. To give an idea of the difficulty involved, the characteristic function for one variate of noise alone is obtained as follows:

$$C = e^{-\frac{a^2}{2}} \int_0^{\infty} v e^{-\frac{v^2}{2} - i\omega v} dv . \quad (102)$$

This is pair 903.3, Campbell and Foster, and may be evaluated directly by completing the square or by forming a differential equation, giving in either case

$$C = 1 - \sqrt{\frac{\pi}{2}} p e^{\frac{p^2}{2}} \operatorname{erfc} \frac{p}{\sqrt{2}} \quad (103)$$

or in terms of ω

$$C = 1 - \omega e^{-\frac{\omega^2}{2}} \left[\int_0^{\omega} e^{\frac{x^2}{2}} dx + i \sqrt{\frac{\pi}{2}} \right] . \quad (104)$$

To raise this expression to the N^{th} power and then obtain the anticharacteristic function is practically hopeless.

The distribution function for the sum of two variates of noise alone is obtainable by use of the convolution theorem, giving

$$dP_2 = \frac{y}{2} e^{-\frac{y^2}{2}} + \frac{\sqrt{\pi}}{2} e^{-\frac{y^2}{4}} \left(\frac{y^2}{2} - 1 \right) \operatorname{erf} \frac{y}{2} dy \quad (105)$$

and the cumulative distribution is also obtainable, giving

$$P_2 = \int_y^\infty f(y) dy = e^{-\frac{y^2}{2}} + \frac{\sqrt{\pi}}{2} y e^{-\frac{y^2}{4}} \operatorname{erf} \frac{y}{2} \quad (106)$$

However, when $N > 2$ there seems to be no closed solution corresponding to Eq.(105) or (106). Since these cases are for noise alone, the signal-plus-noise situation must be attacked by other means.

It turns out that if the moments of the distribution for one variate are known, the moments of the distribution for the sum of N variates may be found directly. Formulae are given, for instance, in Cramer, page 345, ⁽¹⁰⁾ for the first few central moments, which are

$$\mu_2^N = N\mu_2 \quad (107a)$$

$$\mu_3^N = N\mu_3 \quad (107b)$$

$$\mu_4^N = N\mu_4 + 3N(N-1)\mu_2^2 \quad (107c)$$

$$\mu_6^N \approx 10N^3\mu_3^3 + 15N^3\mu_2^3 + 15N^2\mu_2\mu_4 - 45N^2\mu_2^3 \quad (107d)$$

The corresponding coefficients in the Gram-Charlier series then become

$$c_3 = -\frac{\alpha_3}{3!N^{1/2}} \quad (108a)$$

$$c_4 = \frac{\alpha^4 - 3}{4!N} \quad (108b)$$

$$c_6 = \frac{10\alpha_3^2}{6!N} \quad (108c)$$

The α 's in Eqs.(108a-c) are the central standard moments for one variate. Note that, in the square law case, if N is put equal to 1 in Eqs.(85c-f) and the resulting μ 's used in Eqs.(107a-d), the μ 's for N variates are correctly given. If a moment generating function can be found for the case of N variates, then it is immaterial which method is used; but in the case in which such a function is not available, the Eqs.(107) must be used (or some method essentially equivalent).

To handle the linear detector it is now sufficient to find the moments for one variate only. Rice⁽¹⁸⁾, page 107, gives the required expression as

$$\nu_i = 2^{i/2} \Gamma\left(1 + \frac{i}{2}\right) {}_1F_1\left(-\frac{i}{2}, 1, -x\right) . \quad (109)$$

Rice also gives the first two moments as

$$\nu_1 = \sqrt{\frac{\pi}{2}} e^{-\frac{x}{2}} \left[(1+x) I_0\left(\frac{x}{2}\right) + x I_1\left(\frac{x}{2}\right) \right] \quad (110a)$$

$$\nu_2 = 2(1+x) . \quad (110b)$$

To calculate ν_3 one needs to know the function, ${}_1F_1(-3/2, 1, -x)$. This may be obtained by use of the recursion relation

$$a {}_1F_1(a+1, c, z) + (a-c) {}_1F_1(a-1, c, z) = (2a+z-c) {}_1F_1(a, c, z) \quad (111)$$

by putting $a = -\frac{1}{2}$, $c = 1$, $z = -x$. The result is

$$\nu_3 = 2\nu_1(2+x) - \sqrt{\frac{\pi}{2}} e^{-\frac{x}{2}} I_0\left(\frac{x}{2}\right) \quad (110c)$$

also

$$\nu_4 = 4(2+4x+x^2) . \quad (110d)$$

The corresponding central moments are

$$\mu_2 = \sigma^2 = 2(1+x) - \nu_1^2 \quad (112a)$$

$$\mu_3 = 2\nu_1^3 - 2\nu_1(1+2x) - \sqrt{\frac{\pi}{2}} e^{-\frac{x}{2}} I_0\left(\frac{x}{2}\right) \quad (112b)$$

$$\mu_4 = 4(2+4x+x^2) - 3\nu_1^4 - 4\nu_1 \left[(1-x)\nu_1 - \sqrt{\frac{\pi}{2}} e^{-x} I_0\left(\frac{x}{2}\right) \right] \quad (112c)$$

The standard central moments, and then the c 's of Eqs.(108a-c), are directly obtainable from these formulae, though the process is somewhat tedious due to the cumbersome form of Eqs.(112a-c). The functions ν_1 to ν_4 are shown graphically as a function of x in Fig.34.

To obtain the bias level Y_b for the linear detector for $N > 2$, one can use the G.C. series for noise alone. Setting $x = 0$ and $\nu_1 = \sqrt{\pi/2}$ in Eqs.(112a-c) gives

$$\mu_2 = \sigma^2 = 2 - \frac{\pi}{2} = 0.429 \quad (113a)$$

$$\mu_3 = \sqrt{\frac{\pi}{2}}(\pi-3) = 0.1772 \quad (113b)$$

$$\mu_4 = 8 - \frac{3\pi^2}{4} = 0.598 \quad (113c)$$

and

$$a_3 = \frac{\mu_3}{\sigma^3} = 0.632 \quad (114a)$$

$$c_3 = -\frac{0.1053}{N^{1/2}} \quad (114b)$$

and

$$a_4 = \frac{\mu_4}{\sigma^4} = 3.26 \quad (115a)$$

$$c_4 = \frac{0.0108}{N} \quad (115b)$$

$$c_6 = \frac{0.00555}{N} \quad (115c)$$

The cumulative distribution function is now equated to Γ_N , giving

$$\frac{0.693N}{n} = \frac{1}{2} [1 - \phi^{-1}(T)] + \frac{0.1053}{N^{1/2}} \phi^2(T) - \frac{0.0108}{N} \phi^3(T) - \frac{0.00555}{N} \phi^5(T) \quad \dots (116)$$

where

$$T = \frac{Y_b - \nu_1 N}{\sigma \sqrt{N}} = \frac{Y_b - N \sqrt{\frac{\pi}{2}}}{\sqrt{N(2 - \frac{\pi}{2})}}$$

For any given n and N , Y_b may be found from Eq.(116) by trial and error methods. If an approximate value of T is found by neglecting all but the first term in Eq.(116), a more accurate value obtained by Newton's method is

$$T_2 = T_1 - \frac{f(T_1)}{f'(T_1)} \quad (117)$$

It is better, however, to plot Eq.(116) giving n as a function of T and N from which is finally obtained the bias level graph of Fig.35 showing Y_b as a function of n and N for the linear detector.

Since for finding the bias level it is necessary to know the distribution functions only for large values of the argument, it is possible to find an approximate solution valid in this region. Consider a distribution function given by

$$dP = v e^{-\frac{v^2}{2}} dv \quad (117a)$$

for v going from $-\infty$ to $+\infty$. The N^{th} convolution of this function will be nearly the same for large values as if (117a) went only from 0 to ∞ , because the large values in the sum of N variates are most probably produced by addition of large

values of every variate, and for large values (in fact for all positive values) the two distribution functions are identical. The characteristic function of Eq. (117a) is given by pair 710.1 of Campbell and Foster to be

$$C = -\sqrt{2\pi} p e^{\frac{p^2}{2}} . \quad (117b)$$

For the sum of N variates

$$C_N = (-1)^N (2\pi)^{\frac{N}{2}} p^N e^{\frac{NP^2}{2}} . \quad (117c)$$

The probability density function is obtained from pair 740.2 of Campbell and Foster as

$$dP_N \approx \frac{(2\pi)^{\frac{N}{2}-1}}{N^{\frac{N+1}{2}}} e^{-\frac{y^2}{4N}} D_N\left(\frac{y}{\sqrt{N}}\right) dy \quad y \gg 1 \quad (117d)$$

where D_N is the parabolic cylinder function of order N . In terms of the derivative of the error integral as defined in Eq.(62),

$$dP_N \approx \frac{(2\pi)^{\frac{N}{2}}}{N^{\frac{N+1}{2}}} \phi^N\left(\frac{y}{\sqrt{N}}\right) dy \quad y \gg 1 . \quad (117e)$$

Note that for $N = 2$, Eq.(117e) becomes

$$dP_2 \approx \frac{\sqrt{\pi}}{2} e^{-\frac{y^2}{4}} \left(\frac{y^2}{2} - 1\right) . \quad (117f)$$

Referring to Eq.(105), the exact expression for this case, it is seen that Eq.(117f) can be obtained by neglecting the first term and replacing $\operatorname{erf} y/2$ by 1, both of these approximations being very good if $y \gg 1$.

The approximate cumulative distribution is easily obtained from Eq.(117e) by direct integration and gives

$$P_N \approx \left(\frac{2\pi}{N}\right)^{\frac{N}{2}} \phi^{N-1}\left(\frac{Y}{\sqrt{N}}\right) . \quad (117g)$$

The bias level is easily obtainable from this expression by equating it to Γ_N and solving for Y_b by means of the tables of ϕ^i , or by plotting graphs. The method is not very practical for $N > 20$ since suitable tables do not exist.

It is interesting to note that no such approximation as Eq.(117g) is obtainable for the square law case.

Graphs of the probability density functions for signal-plus-noise have been obtained by numerical convolution for some selected cases and are shown in Figs.36 to 41.

RESULTS OF THE LINEAR DETECTOR CALCULATIONS

The difference in results for the linear and square law detectors turns out to be so small that extreme accuracy must be used in the calculations to show the relation in its true form. One such comparison graph was calculated and is shown in Fig.42. Also, in Fig.43 is shown the difference in db in the two cases at $P = 0.50$. The two are identical at $N = 1$, the linear law becomes better by a maximum of 0.11 db at $N = 10$, the two are again equal at $N = 70$, and the square law then becomes better and asymptotically exceeds the linear law by 0.19 db as $N \rightarrow \infty$ having reached 0.16 db at $N = 1000$. These results show conclusively that there is little to choose between the linear law and square law as far as theoretical signal threshold is concerned.

EXPANSIONS IN LAGUERRE SERIES

In certain cases, particularly for low values of N , the Gram-Charlier series may not be the best-suited type of expansion for distribution functions which are zero for all negative values of the amplitude. For low values of N , a suitable expansion for such functions is the following:

$$f(y) = \sum_{i=0}^{i=\infty} a_i e^{-y} y^a L_i^a(y) \quad (118)$$

where $L_i^a(y)$ is the generalized Laguerre polynomial defined by Eq.(82), or by

$$L_i^a(z) = \frac{e^z z^{-a}}{i!} \frac{d^i}{dz^i} (e^{-z} z^{i+a}) \quad (119)$$

The orthogonality relation which makes the expansion possible is

$$\frac{i!}{\Gamma(a+i+1)} \int_0^{\infty} e^{-z} z^a L_i^a(z) L_j^a(z) = \delta_{ij} \quad (120)$$

(see Copson⁽⁴⁾, page 269).

Thus, from Eqs.(118) and (120), the coefficients are determined by

$$a_i = \frac{i!}{\Gamma(\alpha+i+1)} \int_0^\infty L_i^\alpha(y) f(y) dy . \quad (121)$$

Note that Eqs.(118), (120) and (121) are analogous to Eqs.(64), (63) and (65), respectively, for the Gram-Charlier expansion.

Let a new variable $t = y/\beta$. Then

$$f(y) = g(t) = \sum_{i=0}^{\infty} c_i e^{-t} t^\alpha L_i^\alpha(t) \quad (122)$$

where

$$c_i = \frac{i!}{\Gamma(\alpha+i+1)} \int_0^\infty L_i^\alpha(t) g(t) dt = \frac{i!}{\Gamma(\alpha+i+1)} \int_0^\infty L_i^\alpha\left(\frac{y}{\beta}\right) f(y) \frac{dy}{\beta} . \quad (123)$$

The first few Laguerre polynomials are

$$L_0^\alpha(z) = 1 \quad (124a)$$

$$L_1^\alpha(z) = 1 + \alpha - z \quad (124b)$$

$$2L_2^\alpha(z) = (\alpha+1)(\alpha+2) - 2z(\alpha+2) + z^2 \quad (124c)$$

$$6L_3^\alpha(z) = (\alpha+1)(\alpha+2)(\alpha+3) - 3z(\alpha+2)(\alpha+3) + 3z^2(\alpha+3) - z^3 . \quad (124d)$$

Therefore

$$c_1 = \frac{1}{\Gamma(\alpha+2) \cdot \beta} \left[1 + \alpha - \frac{\nu_1}{\beta} \right] \quad (125)$$

$$c_2 = \frac{1}{\Gamma(\alpha+3) \cdot \beta} \left[(\alpha+1)(\alpha+2) - \frac{2\nu_1}{\beta}(\alpha+2) + \frac{\nu_2}{\beta^2} \right] . \quad (126)$$

Since there are two arbitrary constants, α and β , in the expansion of Eq.(122), it is possible to make $c_1 = c_2 = 0$ by a proper choice of α and β . These relations are easily determined by equating Eqs.(125) and (126) to zero and solving simultaneously. The results are

$$\alpha = \frac{\nu_1^2}{\nu_2 - \nu_1^2} - 1 = \frac{\nu_1^2}{\sigma^2} - 1 \quad (127)$$

$$\beta = \frac{\nu_2 - \nu_1^2}{\nu_1} = \frac{\sigma^2}{\nu_1} \quad (128)$$

and

$$c_0 = \frac{1}{\beta \Gamma(\alpha+1)} = \frac{\nu_1}{\sigma^2 \Gamma\left(\frac{\nu_1^2}{\sigma^2}\right)} \quad (129)$$

$$c_1 = c_2 = 0 \quad (130)$$

$$c_3 = \frac{1}{\beta \Gamma(\alpha+4)} \left[\frac{\nu_2}{\beta^2} (\alpha+3) - \frac{\nu_3}{\beta^3} \right] \quad (131)$$

The coefficients past c_0 are so complicated that the whole value of this type of series seems to depend on the fact that the first term alone is often a good approximation. This approximation is

$$dP = \frac{\nu_1}{\sigma^2 \Gamma\left(\frac{\nu_1^2}{\sigma^2}\right)} e^{-\frac{\nu_1 y}{\sigma^2}} \left(\frac{\nu_1 y}{\sigma^2}\right)^{\frac{\nu_1^2}{\sigma^2} - 1} dy \quad (132)$$

and the corresponding cumulative distribution function is

$$P = \int_y^\infty f(y) dy = 1 - I\left[\frac{y}{\sigma}, \frac{\nu_1^2}{\sigma^2} - 1\right] \quad (133)$$

where I is the incomplete gamma function as defined by Eq.(40).

There is a striking analogy between Eq.(132) and the corresponding normal approximation. In both cases the distribution for the sum of N variates is simply obtained by multiplying both ν_1 and σ^2 by N . As $N \rightarrow \infty$, both the normal approximation and Eq.(132) approach the true distribution (and each other). In any particular case, however, the convergence properties of one approximation will be more useful than the other.

In the square law case, for $x = 0$, $\nu_1 = N$ and $\sigma^2 = N$. Substitution of these values in Eq.(132) gives

$$dP = \frac{1}{\Gamma(N)} e^{-y} y^{N-1} dy . \quad (134)$$

Note that this is the same as Eq.(38), the exact expression. Thus in this particular case the first term gives the whole correct result. The third coefficient from Eq.(131) is easily shown to be zero, as all the following coefficients will be.

In the square law case where $x \neq 0$, $\nu_1 = N(1+x)$ and $\sigma^2 = N(1+2x)$. Substitution of these values in Eq.(132) gives

$$dP = \frac{1+x}{(1+2x) \Gamma \left[\frac{N(1+x)^2}{1+2x} \right]} e^{-\left(\frac{1+x}{1+2x}\right)y} \left[\left(\frac{1+x}{1+2x}\right)y \right]^{\frac{N(1+x)^2}{1+2x} - 1} dy \quad (135)$$

and from Eq.(133),

$$P = 1 - I \left[\frac{y}{\sqrt{N(1+2x)}}, \frac{N(1+x)^2}{1+2x} - 1 \right] . \quad (136)$$

A comparison of the particular case $N = 3$, $x = 1$ is shown in Fig.44. Curves are given for the exact distribution function (Eq.47) and the two approximations given by Eqs.(96) and (135).

For the linear case with $x = 0$, $\nu_1 = N\sqrt{\pi/2}$ and $\sigma^2 = N(2-\pi/2)$, the cumulative distribution is, from Eq.(133),

$$P = 1 - I \left[\frac{y}{\sqrt{N(2-\pi/2)}}, \frac{N}{4/\pi-1} - 1 \right] . \quad (137)$$

OTHER SERIES APPROXIMATIONS

It is theoretically possible to develop still other series approximations for the various distribution functions. For instance, it might be thought advantageous to use a sum of terms of the type $y^a e^{-y^2/2}$, particularly in the linear case. While this turns out to be possible, even the first coefficient is so difficult to calculate that the process is impractical.

METHODS OF INTEGRATION INVOLVING SUBTRACTION OF NOISE

Certain practical difficulties arise in maintaining the bias level at the correct value in an electronic detector, particularly if the number of pulses integrated is large. The trouble may arise from fluctuations in amplifier gain, the bias supply, or the noise level itself.

A solution of this problem is to have the gain of the amplifier, or the bias level, or both, controlled by some sort of average value of the noise output. Obviously the time constant of the control device must be neither too long nor too short. One scheme which has been used is to subtract a pulse known to consist of noise only from each possible signal-plus-noise pulse* (see paragraph 3, page 11, No.1). Thus in the absence of a signal, the average value of any number of composite pulses will always be zero, and the required bias level will be comparatively low.

DISTRIBUTION FUNCTIONS FOR COMPOSITE PULSES OF SIGNAL-PLUS-NOISE MINUS NOISE

When a noise pulse is subtracted from each signal-plus-noise pulse, the theoretical distribution functions will be entirely different from previous cases. The square law case is the only one that can be treated in any reasonable fashion. The distribution function for one variate of signal plus noise is given by

$$dP = e^{-x \cdot Y} I_0(2\sqrt{xY}) dY \quad (138)$$

and the characteristic function is

$$C = \frac{e^{-x}}{p+1} e^{\frac{x}{p+1}} \quad (139)$$

* This subtraction can be accomplished by means of a gate which operates at double the repetition frequency. On every other gate only a noise pulse of reversed phase goes through the integrator.

Subtracting a positive noise variate is equivalent to adding a negative noise variate. The distribution function for a negative noise variate is

$$\begin{aligned} dP &= e^Y & Y < 0 \\ &= 0 & Y > 0 \end{aligned} \quad (140)$$

and

$$C = \frac{1}{1-p} \quad (141)$$

To obtain the characteristic function for the sum of a variate from the distributions of Eqs.(138) and (140) it is only necessary to take the product of the characteristic functions given by Eqs.(139) and (141), giving

$$C = \frac{e^{-x}}{1-p^2} e^{\frac{x}{p+1}} \quad (142)$$

This is the characteristic function for one so-called composite pulse. The characteristic function for the sum of N composite pulses is simply

$$C = \frac{e^{-Nx}}{(1-p^2)^N} e^{\frac{Nx}{p+1}} \quad (143)$$

In the case of noise alone ($x = 0$),

$$C = \frac{1}{(1-p^2)^N} \quad (144)$$

and the anticharacteristic function is, by pair 569 Campbell and Foster,

$$dP_N = \frac{1}{\sqrt{\pi}(N-1)!} \left| \frac{Y}{2} \right|^{N-\frac{1}{2}} K_{N-\frac{1}{2}} |Y| dY \quad (145)$$

where $K_{N-\frac{1}{2}}$ is a modified Bessel function of the second kind and is given by the finite series

$$K_{N-\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \sum_{r=0}^{r=N-1} \frac{(N+r-1)!}{r!(N-r-1)!(2z)^r} \quad (146)$$

The cumulative distribution for the sum of N composite noise variates may be found by use of the series (146) and term by term integration. However, for N greater than 3 or 4 the process rapidly becomes impractical.

Again, it is necessary to find moments and proceed by means of Gram-Charlier series. For noise alone, the moments are easily found from Eq.(144) to be

$$\nu_i = \mu_i = 0, \quad i \text{ odd} \quad (147)$$

$$\nu_i = \mu_i = \frac{\left(N + \frac{i}{2} - 1\right)! (i)!}{(N-1)! \left(\frac{i}{2}\right)!}, \quad i \text{ even}$$

in particular,

$$\mu_2 = 2N = \sigma^2 \quad (148)$$

$$\mu_4 = 12N(N+1) \quad (149)$$

$$\mu_6 = 120N(N+1)(N+2) \quad (150)$$

and

$$\alpha_3 = 0 \quad (151)$$

$$\alpha_4 = 3 + \frac{3}{N} \quad (152)$$

$$\alpha_6 = 15 + \frac{45}{N} + \frac{30}{N^2} \quad (153)$$

The only coefficients different from zero in the first six are c_0 and c_4 to the order of $1/N$.

$$c_4 = \frac{1}{8N} \quad (154)$$

Thus

$$dP_N = \frac{dY}{\sqrt{2N}} \left[\phi^0 \left(\frac{Y}{\sqrt{2N}} \right) + \frac{1}{8N} \phi^4 \left(\frac{Y}{\sqrt{2N}} \right) \text{-----} \right] \quad (155)$$

and similar to Eq.(98) is the cumulative distribution

$$P_N = \frac{1}{2} \left[1 - \phi^{-1} \left(\frac{Y}{\sqrt{2N}} \right) \right] - \frac{1}{8N} \phi^3 \left(\frac{Y}{\sqrt{2N}} \right) \text{ ----} \quad (156)$$

The bias number is found by setting this expression equal to Γ_N and plotting Y_b as a function of n and N . Results are given in Fig. 45. In the special case $N = 1$, the cumulative distribution function is simply $e^{-Y/2}$ for $Y > 0$, and the bias number is obtained from this expression rather than from Eq.(156). The anticharacteristic function of the general case, Eq.(143), may be obtained by use of the convolution theorem, pair 202, Campbell and Foster. Let

$$F_1 = \frac{e^{-Nx} e^{\frac{Nx}{p+1}}}{(p+1)^N} \quad (157)$$

$$F_2 = \frac{1}{(1-p)^N} \quad (158)$$

Then from Eq.(37),

$$G_1 = \begin{cases} \left(\frac{y}{Nx} \right)^{\frac{N-1}{2}} e^{-y-Nx} I_{N-1} (2\sqrt{Nxy}) & y > 0 \\ 0 & y < 0 \end{cases} \quad (159)$$

and by pair 525.2, Campbell and Foster,

$$G_2 = \begin{cases} 0 & y > 0 \\ \frac{(-y)^{N-1} e^y}{(N-1)!} & y < 0 \end{cases} \quad (160)$$

Applying the convolution theorem gives

$$dP_N = dY \frac{e^{Y-Nx}}{(N-1)!} \int_Y^\infty \left(\frac{y}{Nx} \right)^{\frac{N-1}{2}} (y-Y)^{N-1} e^{-2y} I_{N-1} (2\sqrt{Nxy}) dy \quad Y > 0 \quad (161)$$

For $Y < 0$, the lower limit of the integral in Eq.(161) is 0 rather than Y .

PROBABILITY DENSITY FUNCTIONS FOR $Y < 0$

To evaluate the integral in Eq. (161) when the lower limit is zero is straightforward but tedious. First one evaluates the integral

$$f(k) = \int_0^{\infty} y^k \left(\frac{y}{Nx}\right)^{\frac{N-1}{2}} e^{-2y} I_{N-1}(2\sqrt{Nxy}) dy \quad (162)$$

by use of characteristic functions in a manner entirely similar to that used in Eqs. (74) to (80). The characteristic function of the function of Eq. (162), with $k = 0$, is

$$C = \frac{e^{\frac{Nx}{p+2}}}{(p+2)^N} \quad (163)$$

and

$$f(k) = \frac{e^{\frac{Nx}{2}} (N+k-1)!}{(N-1)! 2^{N+k}} {}_1F_1\left(-k, N, -\frac{Nx}{2}\right) \quad (164)$$

Then by expanding $(y-Y)^{N-1}$, one obtains the coefficient of

$$y^k = \frac{(N-1)!}{k!} \frac{(-Y)^{N-1-k}}{(N-1-k)!} \quad (165)$$

and from Eqs. (164), (165), and (161),

$$dP_N = \frac{e^{Y-Nx}}{(N-1)!} \sum_{k=0}^{N-1} f(k) \frac{(N-1)!}{k!} \frac{(-Y)^{N-1-k}}{(N-1-k)!} \quad (166)$$

or

$$dP_N = dY \frac{e^{Y-\frac{Nx}{2}}}{(N-1)!} \sum_{k=0}^{N-1} \frac{(N+k-1)!}{(N-k-1)! k! 2^{N+k}} {}_1F_1\left(-k, N, -\frac{Nx}{2}\right) (-Y)^{N-k-1} \quad Y < 0 \quad (167)$$

In terms of Laguerre polynomials, using Eq.(82),

$$dP_N = dY e^{\frac{Y-Nx}{2}} \sum_{k=0}^{N-1} \frac{L_k^{N-1}\left(-\frac{Nx}{2}\right)}{(N-k-1)! 2^{N+k}} (-Y)^{N-k-1} \quad Y < 0 . \quad (168)$$

The first few polynomials are given in Eqs.(124a-d). Some special cases of Eq.(168) are

$$N = 1 \quad dP_1 = dY \frac{e^{\frac{Y-x}{2}}}{2} \quad Y < 0 \quad (169a)$$

$$N = 2 \quad dP_2 = dY \frac{e^{Y-x}}{4} \left(1 + \frac{x}{2} - Y\right) \quad Y < 0 \quad (169b)$$

$$N = 3 \quad dP_3 = dY \frac{e^{\frac{Y-3x}{2}}}{16} \left[3 + 3x + \frac{9x^2}{16} - \left(2 + \frac{3x}{2}\right)Y + Y^2\right] \quad Y < 0 . \quad (169c)$$

The cumulative distributions for $Y < 0$ may easily be obtained by integrating (169a-c). Obviously, the expressions in Eqs.(167) and (168) are practically useful only for small values of N .

PROBABILITY DENSITY FUNCTIONS FOR $Y > 0$

To find a general expression for Eq.(161) giving the distribution function when $y > 0$ is a task of tremendous proportions. Consider, for instance, the special case $N = 1$. Equation (161) becomes

$$dP_1 = dY e^{Y-x} \int_Y^{\infty} e^{-2y} I_0(2\sqrt{xy}) dy \quad Y > 0 . \quad (170)$$

By means of the substitution $y = v^2/4$, this becomes

$$dP_1 = dY \frac{e^{\frac{Y-x}{2}}}{2} \int_{2\sqrt{Y}}^{\infty} e^{-\frac{v^2+x}{2}} I_0(v\sqrt{x}) dv \quad Y > 0 . \quad (171)$$

This can be expressed in terms of the Q function defined by Eq.(16).

$$dP_1 = dY \frac{e^{\frac{Y-x}{2}}}{2} Q(\sqrt{x}, 2\sqrt{Y}) \quad Y > 0 . \quad (172)$$

Eq.(169a) was

$$dP_1 = \frac{e^{\frac{Y-x}{2}}}{2} dY \quad Y < 0 . \quad (169a)$$

Thus, for the case $N = 1$, the whole distribution function is described by Eqs.(169a) and (172). A graph of this function for various values of x is shown in Fig.46. Note that if $x = 0$, $Q(0, 2\sqrt{Y}) = e^{-2Y}$, and Eq.(172) for $Y > 0$ reduces to

$$dP_1 = \frac{e^{-Y}}{2} dY \quad Y > 0 \quad (173)$$

and from Eq.(169a)

$$dP_1 = \frac{e^Y}{2} dY \quad Y < 0 \quad (174)$$

when $x = 0$. Thus over the whole range of Y

$$dP_1 = \frac{e^{-|Y|}}{2} dY \quad (175)$$

which checks Eq.(145) when $N = 1$.

For $N = 2$, Eq.(161) becomes

$$dP_2 = dY e^{Y-2x} \int_Y^\infty \left(\frac{y}{2x}\right)^{\frac{1}{2}} (y-Y) e^{-2y} I_1(2\sqrt{2xy}) dy \quad Y > 0 . \quad (176)$$

This integral may also be evaluated in terms of the Q function. The process requires a large number of integrations by parts and is very time-consuming. The result turns out to be

$$dP_2 = dY \left[\frac{e^{Y-x}}{4} \left(1 + \frac{x}{2} - Y\right) Q(\sqrt{2x}, 2\sqrt{Y}) + \frac{e^{Y-2x}}{4} \left\{ Y I_0(2\sqrt{2xY}) + (1+x) \sqrt{\frac{Y}{2x}} I_1(2\sqrt{2xY}) \right\} \right] \quad Y > 0 . \quad (177)$$

This equation is already so complicated as to be nearly useless. Thus it was not thought worth while to seek a general expression of this type for arbitrary N when $Y > 0$.

Note: If $x = 0$ in Eq.(77), it reduces to

$$dP_2 = dY \frac{e^{-Y}}{4} (1+Y) \quad Y > 0 \quad (178)$$

which may also be obtained from Eq.(169b) by substituting $-Y$ for Y .

CUMULATIVE DISTRIBUTION FUNCTIONS

The effort in Eqs.(161) to (178) has been concerned with obtaining the probability density function for N variates of signal-plus-noise minus noise. To find the cumulative distribution functions exactly is difficult, especially for Y positive.

A case which can be solved, however, is that for $N = 1$. For Y negative the answer is simply obtained from Eq.(169a) and is

$$P_1 = 1 - \frac{e^{Y-\frac{x}{2}}}{2} \quad Y < 0 \quad (178a)$$

For Y positive, using the result of Eq.(172),

$$P_1 = \frac{e^{-\frac{x}{2}}}{2} \int_Y^{\infty} e^y Q(\sqrt{x}, 2\sqrt{y}) dy \quad Y > 0 \quad (178b)$$

Since the value of P_1 at $Y = 0$ is, from Eq.(178a), $1 - e^{-\frac{x}{2}}/2$, Eq.(178b) may be re-written as

$$P_1 = 1 - \frac{e^{-\frac{x}{2}}}{2} - \frac{e^{-\frac{x}{2}}}{2} \int_0^Y e^y Q(\sqrt{x}, 2\sqrt{y}) dy \quad (178c)$$

but from the definition of Q in Eq.(16),

$$Q(\sqrt{x}, 2\sqrt{y}) = \int_{2\sqrt{y}}^{\infty} v e^{-\frac{v^2+x}{2}} I_0(v\sqrt{x}) dv = 2e^{-\frac{x}{2}} \int_y^{\infty} e^{-2z} I_0(2\sqrt{xz}) dz \quad (178d)$$

Replacing Q by its defining integral in Eq.(178c) gives

$$P_1 = 1 - \frac{e^{-\frac{x}{2}}}{2} - e^{-x} \int_0^Y dy e^y \int_y^\infty e^{-2z} I_0(2\sqrt{xz}) dz . \quad (178e)$$

Integration by parts is now used, letting

$$\left. \begin{aligned} u &= \int_y^\infty e^{-2z} I_0(2\sqrt{xz}) dz \\ dv &= e^y dy \\ du &= -e^{-2y} I_0(2\sqrt{xy}) dy \\ v &= e^y \end{aligned} \right\} \quad (178f)$$

$$|uv|_0^Y = e^Y \int_Y^\infty e^{-2z} I_0(2\sqrt{xy}) dz - \frac{e^{\frac{x}{2}}}{2} \quad (178g)$$

$$= \frac{e^{\frac{Y+x}{2}}}{2} Q(\sqrt{x}, 2\sqrt{Y}) - \frac{e^{\frac{x}{2}}}{2} . \quad (178h)$$

Thus

$$P_1 = 1 - \frac{e^{\frac{Y-x}{2}}}{2} Q(\sqrt{x}, 2\sqrt{Y}) + e^{-x} \int_0^Y v du \quad (178i)$$

or

$$P_1 = 1 - \frac{e^{\frac{Y-x}{2}}}{2} Q(\sqrt{x}, 2\sqrt{Y}) - e^{-x} \int_0^Y e^{-y} I_0(2\sqrt{xy}) dy . \quad (178j)$$

The integral term in Eq.(178j) is just $1 - Q(\sqrt{2x}, \sqrt{2Y})$, and the final result is

$$P_1 = Q(\sqrt{2x}, \sqrt{2Y}) - \frac{e^{\frac{Y-x}{2}}}{2} Q(\sqrt{x}, 2\sqrt{Y}) \quad Y > 0 . \quad (178k)$$

For $x = 0$, $Q(0, \beta) = e^{\frac{\beta^2}{2}}$, and

$$P_1 = e^{-Y} - \frac{e^Y}{2} (e^{-2Y}) = \frac{e^{-Y}}{2} \quad (178l)$$

agreeing, as it should, with the result obtained from Eq.(145) by letting $N = 1$ and integrating. For $Y = 0$, $Q(\alpha, 0) = 1$, and

$$P_1 = 1 - \frac{e^{-\frac{x}{2}}}{2} \quad (178m)$$

agreeing with Eq.(178a) when $Y = 0$.

The bias number for use with Eq.(178k) is obtained by

$$\Gamma_1 = \frac{0.693}{n} = \frac{e^{-Y_b}}{2} \quad (178n)$$

or

$$Y_b = 2.30 \log_{10} n - 0.327. \quad (178o)$$

In Fig. 47 is shown a graph comparing Eq.(178k) with Eq.(23) for $n = 10^6$, where P is plotted as a function of x .

Though it might be possible to calculate the cumulative distributions for $N > 1$ by a method similar to that used for $N = 1$, it would be very tedious. Therefore resort is made to Gram-Charlier series, as before. The moments are directly obtainable from the characteristic function given in Eq.(143),

$$\nu_i = (-1)^i \frac{d^i}{dp^i} \left| \frac{e^{-Nx} e^{\frac{Nx}{p+1}}}{(1-p^2)^N} \right|_{p=0} \quad (179)$$

There seems to be no readily obtainable expression for ν_i in closed form. The first six moments obtained directly from Eq.(179) are:

$$\nu_1 = Nx \quad (180a)$$

$$\nu_2 = (Nx)^2 + 2Nx + 2N \quad (180b)$$

$$\nu_3 = (Nx)^3 + 6(Nx)^2 + 6Nx(N+1) \quad (180c)$$

$$\nu_4 = (Nx)^4 + 12(Nx)^3 + 12(Nx)^2(N+3) + 24(N+1) + 12N(N+1) \quad (180d)$$

$$\nu_5 = (Nx)^5 + 20(Nx)^4 + 20(Nx)^3(N+6) + 120(Nx)^2(N+2) + 60Nx(N+1)(N+2). \quad (180e)$$

$$\begin{aligned}
\nu_6 = & (Nx)^6 + 30(Nx)^5 + 30(Nx)^4(N+10) + 120(Nx)^3(3N+10) \\
& + 180(Nx)^2(N+1)(N+6) + 360Nx(N+1)(N+2) \\
& + 120N(N+1)(N+2) .
\end{aligned} \tag{180f}$$

The corresponding central moments are:

$$\mu_2 = 2Nx + 2N = 2N(1+x) = \sigma^2 \tag{181a}$$

$$\mu_3 = 6Nx \tag{181b}$$

$$\mu_4 = 12(Nx)^2 + 24Nx(N+1) + 12N(N+1) \tag{181c}$$

$$\mu_6 = 120(Nx)^3 + 360(Nx)^2(N+3) + 360Nx(N+1)(N+2) + 120N(N+1)(N+2) . \tag{181d}$$

The central standard moments are:

$$\alpha_3 = \frac{3x}{\sqrt{2N}(1+x)^{\frac{3}{2}}} \tag{182a}$$

$$\alpha_4 = 3 + \frac{3(1+2x)}{N(1+x)^2} \tag{182b}$$

$$\alpha_6 \approx 15 + \frac{45(3x^2+3x+1)}{N(1+x)^3} . \tag{182c}$$

The coefficients of the series are, from Eq.(71):

$$c_3 = - \frac{x}{2\sqrt{2N}(1+x)^{\frac{3}{2}}} \tag{183a}$$

$$c_4 = \frac{1+2x}{8N(1+x)^2} \tag{183b}$$

$$c_6 = \frac{x^2}{16N(1+x)^3} \tag{183c}$$

The Gram-Charlier series for the probability density function is given by Eq.(93) where

$$t = \frac{y - \nu_1}{\sigma}, \quad \nu_1 = Nx, \quad \sigma = \sqrt{2N(1+x)} \quad (184)$$

and the cumulative distribution is given by Eqs.(98) and (99). Figures 53 and 54, No.1, showing the comparison between the ordinary case and the composite case, were computed using the Gram-Charlier series developed above. There appears to be no significant difference in the probabilities of detection for N between 1 and 10. For N between 100 and 1,000, the composite case gives an effective signal-to-noise ratio about 1 db lower than the ordinary case.

ANOTHER APPROACH TO THE DETECTION CRITERIA-PROBABILITY THAT SIGNAL- PLUS-NOISE EXCEEDS NOISE ALONE

The method of setting a bias level and calling any signal-plus-noise or noise alone which exceeds this level a signal is not the only possible way of defining detection. Another method is based on asking what is the probability that any given signal will be larger than any noise pulse during a given interval of time⁽¹¹⁾. The interval of time taken would logically be the false alarm time, as defined previously. In this time there will be $n/N = n'$ independent groups of noise pulses. If the probability that a single integrated group of signal-plus-noise pulses exceeds a single group of noise pulses is called $P_1(x, N)$, then the probability that the group of signal-plus-noise pulses exceeds *all* of the n' groups of noise pulses is simply

$$P = [P_1(x, N)]^{n'} \quad (185)$$

This probability is a little difficult to interpret properly. It means that if during the false alarm time a signal of strength x appears, it will have this probability of being larger than any noise pulse group appearing during the same time. The difficulty is how to pick out the largest signal over a period of time, and what to do when many signals are present. These are reasons why the earlier detection criteria are thought to be superior, since they provide clear answers for the above questions. The criteria presented above may be of special value, however, when a target is *known* to be present. Such is the case when a target is being automatically tracked, and one wishes to calculate the probability that it will be subsequently lost due to the noise exceeding the signal.

The probability density function for N signal-plus-noise pulses minus N noise pulses has been indicated in Eq.(161).

To obtain the probability that the sum of N signal-plus-noise pulses will be greater than N noise pulses it is only necessary to integrate Eq.(161) from 0 to ∞ . It will be easier to obtain the probability that N noise pulses exceed N signal-

plus-noise pulses, however, since this requires the integral from $-\infty$ to 0, and an expression is available for $Y < 0$ in Eq.(167). Thus

$$P_{N>S+N} = \frac{e^{-\frac{Nx}{2}}}{(N-1)!} \int_{-\infty}^0 dY e^Y \sum_{k=0}^{k=N-1} \frac{(N+k-1)! {}_1F_1\left(-k, N, -\frac{Nx}{2}\right)}{(N-k-1)! k! 2^{N+k}} (-Y)^{N-k-1} \quad (186)$$

Now one substitutes z for $-Y$ and interchanges the summation and integration signs, obtaining

$$P = \frac{e^{-\frac{Nx}{2}}}{(N-1)!} \sum_{k=0}^{k=N-1} \left[\frac{(N+k-1)! {}_1F_1\left(-k, N, -\frac{Nx}{2}\right)}{(N-k-1)! k! 2^{N+k}} \int_0^{\infty} e^{-z} z^{N-k-1} dz \right] \quad (187)$$

The integral is simply $(N-k-1)!$, and therefore

$$P = \frac{e^{-\frac{Nx}{2}}}{2} \sum_{k=0}^{k=N-1} \left[\frac{(N+k-1)!}{(N-1)! k! 2^{N+k-1}} {}_1F_1\left(-k, N, -\frac{Nx}{2}\right) \right] \quad (188)$$

Or in terms of Laguerre polynomials, using Eq.(82),

$$P = \frac{e^{-\frac{Nx}{2}}}{2} \sum_{k=0}^{k=N-1} 2^{1-N-k} L_k^{N-1}\left(-\frac{Nx}{2}\right) \quad (189)$$

From Eq.(188) a more convenient form may be obtained by introducing a dummy index i and interchanging summation signs, leading eventually to

$$P = \frac{e^{-\frac{Nx}{2}}}{2} \sum_{i=0}^{i=N-1} \left[\frac{\left(\frac{Nx}{2}\right)^i}{i! (N+i-1)!} \sum_{k=i}^{k=N-1} \frac{(N+k-1)!}{(k-i)! 2^{N+k-1}} \right] \quad (190)$$

The outside summation in Eq.(190) is obviously a polynomial in x of the N -1th degree and with N terms. It is rather curious to note that if one puts $i = 0$ in Eq.(190), the following identity results:

$$2^{N-1} \equiv \sum_{k=0}^{k=N-1} \frac{(N+k-1)!}{(N-1)! k! 2^k} \quad (191)$$

In other words, the constant term in the polynomial is always unity.

The first few cases for low values of N are:

$$P_1 = \frac{1}{2} e^{-\frac{x}{2}} \quad (192a)$$

$$P_2 = \frac{1}{2} e^{-x} \left(1 + \frac{x}{4}\right) \quad (192b)$$

$$P_3 = \frac{1}{2} e^{-\frac{3x}{2}} \left(1 + \frac{9x}{16} + \frac{9x^2}{128}\right) \quad (192c)$$

$$P_4 = \frac{1}{2} e^{-2x} \left(1 + \frac{29x}{32} + \frac{x^2}{4} + \frac{x^3}{48}\right) \quad (192d)$$

$$P_5 = \frac{1}{2} e^{-\frac{5x}{2}} \left(1 + \frac{325x}{256} + \frac{575x^2}{1024} + \frac{4375x^3}{6144} + \frac{625x^4}{32,768}\right) \quad (192e)$$

Obviously for N very large, these expressions rapidly become useless, and it is necessary to use the Gram-Charlier series of Eqs.(184) and (98). The lower limit Y is replaced by zero, giving for the series

$$P = \frac{1}{2} \left[1 - \phi^{-1}(T)\right] + c_3 \phi^2(T) - c_4 \phi^3(T) - c_6 \phi^5(T) \dots \quad (193)$$

where

$$T = \frac{\nu_1}{\sigma} = x \sqrt{\frac{N}{2(1+x)}} \quad (194)$$

and c_3 , c_4 and c_6 are given by Eqs.(183a-c). A graph of P as a function of x and N is shown in Fig.48. For very small values of P , more terms may be necessary in the series of Eq.(193).

USE OF CUMULANTS IN OBTAINING GRAM-CHARLIER SERIES COEFFICIENTS

It is often much simpler to obtain the cumulants for a given distribution function rather than the various moments. The cumulants may be defined by

$$\kappa_i = (-1)^i \left(\frac{d^i}{dp^i} \log_e C \right)_{p=0} \quad (195)$$

where C is the characteristic function of the given probability density function (see pages 61-65 of Kendall⁽⁶⁾). The cumulants, except the first, are invariant with respect to a change of origin. Also, for the distribution of the sum of N variates, it is only necessary to multiply every cumulant by N , as is evident from the defining Eq.(195). The coefficients of the Gram-Charlier series in terms of cumulants are given on page 149 of Kendall. The cumulants in standard measure may be defined as

$$K_i = \frac{\kappa_i}{\sigma^i} . \quad (196)$$

In terms of standard cumulants, the coefficients of the series are:

$$c_0 = 1 \quad c_1 = c_2 = 0 \quad (197a)$$

$$c_3 = \frac{K_3}{3!} \quad (197b)$$

$$c_4 = \frac{K_4}{4!} \quad (197c)$$

$$c_6 = \frac{1}{6!} (K_6 + 10K_3^2) . \quad (197d)$$

The first term in Eq.(197d), K_6 , is omitted in the 0,3,4,6 approximation.

Consider the square law case where, from Eq.(35),

$$C = \frac{e^{-x} e^{\frac{x}{p+1}}}{p+1} \quad (198)$$

$$\log_e C = -x + \frac{x}{p+1} - \ln(p+1) . \quad (199)$$

From Eq.(195),

$$\kappa_i = (i-1)!(ix+1) \quad i \neq 1 . \quad (200)$$

For N variates,

$$\kappa_i = N(i-1)!(ix+1) . \quad (201)$$

and

$$K_i = \frac{\kappa_i}{\sigma^i} = \frac{(i-1)!(ix+1)}{N^{\frac{i}{2}-1} (2x+1)^{\frac{i}{2}}} . \quad (202)$$

In particular,

$$K_3 = \frac{2(3x+1)}{N^{1/2}(2x+1)^{\frac{3}{2}}} \quad (203a)$$

$$K_4 = \frac{6(4x+1)}{N(2x+1)^2} \quad (203b)$$

and it is at once evident that c_3 , c_4 , and c_6 obtained from Eqs.(197b-d) are identical to the values given by Eqs.(92c-e) by means of a much longer process.

In the case of a composite pulse of signal-plus-noise minus noise, the characteristic function is given by Eq.(142) and

$$\log_e C = -x + \frac{x}{p+1} - \log_e (1-p^2) \quad (204)$$

Again by means of Eq.(195) it is easy to derive, for N variates,

$$\begin{aligned} \kappa_i &= N(i-1)! [ix+2] & i \text{ even} \\ &= N(i-1)! (ix) & i \text{ odd, } \neq 1 \end{aligned} \quad (205)$$

or

$$\kappa_i = N(i-1)! [ix+1+(-1)^i] \quad (206)$$

and

$$K_i = \frac{\kappa_i}{\sigma^i} = \frac{(i-1)! [ix+1+(-1)^i]}{N^{\frac{i}{2}-1} [2(x+1)]^{\frac{i}{2}}} \quad (207)$$

Special cases are:

$$K_3 = \frac{6x}{N^{1/2} [2(x+1)]^{\frac{3}{2}}} \quad (208a)$$

$$K_4 = \frac{3(2x+1)}{N(x+1)^2} \quad (208b)$$

and again by Eqs.(197b-d) the coefficients are seen to be the same as given by Eqs.(183a-c).

In a case such as the linear one where the characteristic function cannot be obtained, the cumulants are still useful and may be found from the moments ν_i by means of the formulae at the bottom of page 63 of Kendall. The first few are:

$$\kappa_1 = \nu_1 \quad (209a)$$

$$\kappa_2 = \nu_2 - \nu_1^2 \quad (209b)$$

$$\kappa_3 = \nu_3 - 3\nu_2\nu_1 + 2\nu_1^3 \quad (209c)$$

$$\kappa_4 = \nu_4 - 4\nu_3\nu_1 - 3\nu_2^2 + 12\nu_2\nu_1^2 - 6\nu_1^4 . \quad (209d)$$

The κ_i are now obtained by multiplying by N and dividing by σ^i . The coefficients are then obtained as before by Eqs.(197a-d).

BEST POSSIBLE DETECTOR LAW

It is of considerable importance to know whether there may be some detector law which will give results which are appreciably better than the linear or square law cases which have already been considered.

The problem may be stated as follows:

These are available N samples

$$v_1, v_2 \text{ ---- } v_N$$

which, it is assumed, are known to have come from either the distribution

$$dP_1 = v e^{-\frac{v^2}{2}} dv \quad (210)$$

or the distribution

$$dP_2 = v e^{-\frac{v^2+a^2}{2}} I_0(av) dv \quad (211)$$

the former being the distribution of the envelope of noise alone, and the latter the distribution of signal-plus-noise.

The probability that all of the variates $v_1 \text{ ---- } v_N$ came from the second distribution is simply

$$dP_{N2} = dP_2(v_1)dP_2(v_2) \text{ ---- } dP_2(v_N) \quad (212)$$

whereas the probability that they all came from the first distribution is

$$dP_{N1} = dP_1(v_1)dP_1(v_2) \dots dP_1(v_N) \quad (213)$$

The ratio of dP_{N2} to dP_{N1} is the best measure of the likelihood that all the variates came from the signal-plus-noise distribution. It can be shown that any monotonic function of this ratio gives an equally good significance test. One arbitrarily picks a constant which the ratio must exceed to say that it shows that the variates came from the signal-plus-noise distribution. *This constant determines the false alarm time.*

Taking the ratio of Eq.(213) to Eq.(212) and substituting values from Eqs.(210) and (211) gives

$$\frac{dP_{N2}}{dP_{N1}} = \frac{\prod_{i=1}^{i=N} v_i e^{-\frac{v_i^2 + a^2}{2}} I_0(av_i)}{\prod_{i=1}^{i=N} v_i e^{-\frac{v_i^2}{2}}} \quad (214)$$

or

$$e^{-\frac{a^2}{2}} \prod_{i=1}^{i=N} I_0(av_i) \geq \lambda \quad (215)$$

where λ is the constant which determines the false alarm time.

Taking the log of both sides of Eq.(215) gives

$$\sum_{i=0}^{i=N} \log_e I_0(av_i) \geq \log_e \lambda + \frac{a^2}{2} \quad (216)$$

Note that *nothing* has been said in the foregoing discussion about integration. Now, however, Eq.(216) says that the best thing to do is take the log of I_0 of each variate, add these functions for each variate, and require the sum to exceed a certain value. Clearly this calls for a detector and integrator which has the combined law

$$y = \log I_0(av) \quad (217)$$

The meaning of this result is really quite remarkable (at least to one who is not a statistician). It says, in effect, that by having the sum only of N variates which have been subjected to the law $y = \log I_0(av)$, one has as much useful information as if the individual values of each of the variates were known (as far as determining to which distribution the variates belong)*.

* If the two distribution functions to be distinguished are normal, then the simple sum of the N variates, or the mean, is the best criterion. In other words, a linear law would be the best if the envelopes of noise and signal-plus-noise were normally distributed.

Suppose that the signal strength is very small (which would make N large for any reasonable probability of detection). Then $I_0(av) \approx 1 + a^2v^2/4$ and

$$y = \log I_0(av) \approx \log \left(1 + \frac{a^2v^2}{4}\right) \approx \frac{a^2v^2}{4} \quad (218)$$

In this case, the square law is seen to be the best possible choice. If, on the other hand, the signal strength is large, $I_0(av) \approx e^{av}/\sqrt{2\pi av}$ and

$$y = \log I_0(av) \approx \log \frac{e^{av}}{\sqrt{2\pi av}} \approx av - \frac{1}{2} \log 2\pi av \approx av \quad (219)$$

Thus, for large signals (usually small N) the linear law is best.

It should be pointed out that the results for the two extreme cases, square and linear law, are not very different (see Fig.42), and in practice a linear detector would usually be preferred on account of its relative immunity to saturation by large signals.

In the case of a human operator it is difficult to say what law is used in the process of integration. Thus if a linear detector were used in the receiver, it is conceivable that the operator might mentally take the sum of the squares in his integration process, with a net over-all square law effect.

SIGNAL-PLUS-NOISE MINUS NOISE - LINEAR LAW

This case is of special interest because of the method which must be used in obtaining the solution. Since the characteristic function for the linear case cannot be found, it is necessary to determine the moments for a composite variate directly from the moments for the signal-plus-noise distribution and those for the noise distribution alone.

Using a double subscript notation, in which the first index represents the number of the distribution function and the second index represents the order of the moment, the following formulae can be derived at once by successive differentiations of the product of the characteristic functions of the individual distribution functions:

$$\nu_1 = \nu_{11} + \nu_{12} \quad (220a)$$

$$\nu_2 = \nu_{12} + \nu_{22} + 2\nu_{11}\nu_{22} \quad (220b)$$

$$\nu_3 = \nu_{23} + \nu_{13} + 3(\nu_{11}\nu_{22} + \nu_{12}\nu_{21}) \quad (220c)$$

$$\nu_4 = \nu_{14} + \nu_{24} + 4(\nu_{11}\nu_{23} + \nu_{21}\nu_{13}) + 6\nu_{12}\nu_{22} \quad (220d)$$

$$\nu_6 = \nu_{16} + \nu_{26} + 6(\nu_{25}\nu_{11} + \nu_{15}\nu_{21}) + 15(\nu_{12}\nu_{24} + \nu_{22}\nu_{14}) + 20\nu_{13}\nu_{23} \quad (220e)$$

The first set of moments are those for one variate of signal plus noise given by Eqs.(109) and (110a-d). The second set of moments are for one *negative* variate of noise alone. These are simply obtained from the first set of moments by putting $x = 0$ and multiplying the odd moments by -1.

The details will not be given, since the results bear the same relation in general to the square law case as they do when a noise variate is not subtracted from each signal-plus-noise variate.

USE OF SO-CALLED DETECTION CRITERIA

Lawson and Uhlenbeck have made use of a quantity which is the shift in average value of a distribution of signal-plus-noise from that of noise alone divided by the standard deviation of noise alone, which they call the detection criterion. In symbolic form

$$k = \frac{\nu_{1S+N} - \nu_{1N}}{\sigma_N} \quad (221)$$

This quantity is also called the deflection criterion, and it is implied that it must be of the order of unity or greater to have a reasonable probability of detection.

For the square law detector, using the results of Eqs.(81a-b) and (85c), the criterion becomes

$$k = x\sqrt{N} \quad (222)$$

and for the linear detector

$$k = \frac{x\sqrt{N}}{2\sqrt{\frac{4}{\pi}} - 1} = 0.957x\sqrt{N} \quad (223)$$

assuming x to be small.

The object of these criteria is to show the variation in necessary signal-to-noise ratio as a function of the number of pulses integrated. The results for k in Eqs.(222) and (223) may be derived rigorously from the basic distribution equations if the central limit theorem is assumed to hold and for probability of detection equal to 0.50.

However, it is found from the actual results presented in No.1, Figs.1-50, that the square root of N law given by the detection criteria is not closely followed, even for N as large as 1,000. If a law of the form

$$k = xN^\theta \quad (224)$$

is assumed, the exponent θ may be obtained from the data of Figs.1-50, No.1. The results are given in Figs.55 and 56, No.1. It is seen that θ goes from 1.0 at $N = 1$ to around 0.75 at $N = 1,000$. As pointed out earlier (page 28), $\theta = 1$ for coherent integration.

It has been said that the $N^{1/2}$ law seems to fit observed data fairly well. It is the belief of the author that this is a coincidence that arises from the fact that the losses due to nonlinear integration by cathode ray tubes, and human operator losses, tend to just about equal the difference between N^θ and $N^{1/2}$, so that the $N^{1/2}$ law actually seems to fit the *observed* data.

It is rather interesting to note that if the detector law is assumed to be of the form $y = v^n$, the detection criterion turns out to be

$$k = \frac{n \binom{n-1}{2}}{2\sqrt{n! - \binom{n-1}{2}^2}} x\sqrt{N} . \quad (225)$$

A graph of this function shows a very broad maximum of 1 at $n = 2$. Thus this is a special case, showing that for large N the square law is the best of the particular class of functions v^n . This is not as general as the proof on page 56 which shows that the square law is the best of *all possible functions* for small x .

COLLAPSING LOSS - INTEGRATION OF GREATER NUMBER OF NOISE VARIATES THAN OF SIGNAL-PLUS-NOISE VARIATES

In many radar applications, an additional number of noise variates are integrated along with a given number of signal-plus-noise variates. Such is the case when three-dimensional data are compressed onto a two-dimensional presentation, or with a C scope where range is not shown. The loss so occasioned is called a collapsing loss⁽³⁹⁾. An effect of the same kind is caused if the spot of a cathode ray tube indicator moves less than its diameter in a pulse length⁽¹¹⁾. Again, if the video bandwidth is narrow compared with the IF bandwidth, the same sort of thing happens. All three effects are handled by assuming a given collapsing ratio, ρ , which is defined by

$$\rho = \frac{M+N}{N} \quad (226)$$

where

N = number of signal-plus-noise variates integrated

M = number of effective additional noise variates integrated.

In the case of loss caused by low writing speed of the cathode ray beam, the effective collapsing ratio is given approximately by

$$\rho_{eff} = \frac{d+s\tau}{s\tau} \quad (227)$$

where

d = spot diameter

s = writing speed

τ = pulse length.

Where the loss is caused by a video amplifier, the equivalent defining equation is

$$\rho_{eff} = \frac{B_{if} + B_v}{B_v} \quad (228)$$

where

B_{if} = IF bandwidth (or total combined RF and IF bandwidth where RF amplification is used)

B_v = video bandwidth.

Mathematically, the treatment necessary to take account of M extra noise variates is rather simple. It is only necessary to multiply the characteristic function for N signal-plus-noise variates by the characteristic function for M noise-alone variates. In the square law case, this results in

$$C_N = \frac{e^{-Nx} \frac{Nx}{e^{\rho+1}}}{(p+1)^{N+M}} = \frac{e^{-N\rho(\frac{x}{\rho})} \frac{N\rho(\frac{x}{\rho})}{e^{\rho+1}}}{(p+1)^{N\rho}} \quad (229)$$

It is apparent, by comparison with Eq.(36), that the results obtained for $\rho = 1$ can be used directly to obtain results for any ρ .

Care must be taken in obtaining the bias level, however. Without the M extra noise variates, the relation $n' = n/N$ is used to find the required signal-to-noise ratio, x . With the added noise variates, the number of groups of pulses integrated may or may not remain the same. In the case of *video mixing*, where the output of two independent radars is superimposed on the same indicator, the number of groups of pulses integrated is constant, which means that n' is constant.

In the other cases where the loss is caused by narrow video amplifiers, collapsing of coordinates, or slow writing speed, the number of independent groups of pulses integrated is reduced by the factor ρ_{eff} so that n remains constant as is easily seen from the equations

$$n = n'N \quad (\text{no loss}) \quad (230)$$

$$n = (\rho n')(M+N) = n'N \quad (\text{with loss}). \quad (231)$$

The collapsing loss is defined as

$$L_c = 10 \log_{10} \frac{x_2}{x_1} \quad (232)$$

where x_2 is the required signal-to-noise ratio with M extra noise variates, and x_1 is the signal-to-noise ratio required with no extra noise variates, such that

the probability of detection is the same in both cases. This fixed probability level will usually be taken as 0.90.

The procedure, after finding x_1 , is to get the required bias from either Fig. 8 or 9, depending on whether n or n' is held constant, using ρN as the number of variates. From the cumulative distribution functions graphed in Figs. 13 to 32, the value of x_2 is found by multiplying the finding x for ρN variates to give $P = 0.90$ and multiplying this value of x by ρ . The reason for multiplying by ρ is apparent on referring to Eq.(229).

The results of the calculation are shown in Figs. 49 to 52 where L is plotted as a function of N for $P = 0.90$ and $N = 10^6$. Also given are curves of θ_c defined by

$$\frac{x_2}{x_1} = N^{\theta_c} \quad (233)$$

It has commonly been said that θ_c should be $1/2^{(36)}, (36)$. This statement is sometimes derived from the detection criterion given on page 58.

From Fig. 57 it is seen that if n' is constant, θ does approach $1/2$ as $N \rightarrow \infty$. However, θ is much smaller for reasonably small N . In the case of n constant, the square root law is not even approached as an asymptote.

It was found that the values of L and θ are only slightly dependent on the original values of n and P .

ANTENNA BEAM SHAPE LOSS

It has so far been assumed that the antenna pattern was flat over the half-power beamwidth and zero elsewhere. In any practical case the beam shape may usually be approximated by a Gaussian curve which will hold fairly well out to \pm the beamwidth from the point of maximum gain. In the case of a searchlighting antenna, the returned pulses will all fall at the same place in the beam, and if this does not happen to fall at the maximum of the beam, the loss may easily be taken into account by modifying the expression for gain used in Eq.(9), No.1 for calculating R_0 such that

$$G = G_{\max} e^{-4 \ln 2 \left(\frac{\theta_a^2}{B_a^2} + \frac{\theta_e^2}{B_e^2} \right)} \quad (234)$$

where

- θ_a = azimuth angle between target and antenna axis
- θ_e = elevation angle between target and antenna axis
- B_a = half-power azimuth beamwidth
- B_e = half-power elevation beamwidth

If the antenna is scanning, the problem is entirely changed because the successive returned pulses will be of different magnitude. It is obvious that as the antenna scans past a target, pulses should be integrated out to some point where the principle of diminishing returns sets in. It is not too difficult to determine this point and to calculate the loss occasioned due to the beam shape as compared with the ideal case⁽⁴³⁾. A complete treatment which covers the general case of delay

of the received pulse relative to the transmitted pulse, off axis in elevation while scanning in azimuth, and random orientation of the pulse pattern relative to the antenna pattern is quite involved. However, the solution of some special cases has shown the general character to be expected of the results.

The integration of pulses should be carried to about 1.1 times the half-power beamwidth. This figure is practically independent of the signal strength (range) and the number of pulses per half-power beamwidth. When the optimum number of pulses are integrated there will be an average loss over the ideal case which assumes constant gain between the half-power points. This loss is in the neighborhood of 1.5 db and does not depend much on signal strength or number of pulses per half-power beamwidth. Since this loss is so small it was not considered worth while to reproduce all the detailed calculations here.

It should be mentioned that special care is necessary when one considers rates of antenna scanning so fast that about only 1 hit per beamwidth is obtained. In this case it may be expedient to make the receiving antenna lag the transmitting antenna to compensate for the time of travel of the pulse, or to step-scan, that is, move the antenna in discrete steps rather than continuously.

In order to calculate the probability of detection in any case where the successive returned pulses have different signal strengths, it is necessary to obtain the over-all characteristic function by multiplying the characteristic functions for each pulse. Using this method it is not difficult to work out the needed results in any particular case.

LIMITING LOSS

If limiting occurs anywhere in the receiver, the probability of detection will be lowered, everything else being held constant. The video amplifier is the first place where limiting will probably occur. Let the limiting ratio be defined as the ratio of the limit level to the R.M.S. noise level. Limiting can then be represented mathematically by replacing the probability density function at the detector output by an equivalent function below the limit level, and a delta function at the limit level having an area equal to all of the area of the original function to the right of the limit level. The moments can be calculated for these new functions (noise alone and signal-plus-noise), and the probability of detection found by use of the Gram-Charlier series as usual. The calculations are quite tedious and will not be reproduced here. The main conclusions are that if the number of pulses integrated is large, the limiting loss is only a fraction of a db if the limiting ratio is as large as 2 or 3, but if only one or two pulses are integrated the limiting ratio must be in the neighborhood of 10 to prevent a serious loss.

Limiting in the output of the integrator can also cause a loss, but this loss is small compared to the loss caused by limiting of the individual pulses in most practical cases.

EFFECT OF SIGNAL INJECTION ON PROBABILITY OF DETECTION

It has been proposed that the minimum detectable signal can be decreased by the injection of an RF or IF carrier voltage that adds linearly to the received

echo and the receiver noise⁽⁶¹⁾. The theory is that the total signal will then be large compared with the noise, and thus the so-called modulation suppression that occurs in the process of detection with small signals will be eliminated.

In such a process, the coherence of the injected signal with the received echo must be taken into account. If the target is moving, then the successive received pulses may be considered to be random in phase, so that the injected signal will necessarily be noncoherent with the echo. Analysis has shown that in this case the probability of detection decreases continuously as the magnitude of the injected signal increases, assuming a linear or square law detector. However, it can be shown that the best possible detector law starts to change radically as soon as the injected signal strength becomes comparable to noise. The analysis of probability of detection when the detector function is altered to take into account the injected signal has not been completed. Preliminary estimates indicate that there will be only a small decrease in sensitivity in this case.

It might be imagined that coherence could be obtained in a system using only one hit per target but having, say, 20 separate receiver channels with 20 separate injection oscillators having phases spaced 12 degrees apart. Thus, the return echo would be nearly coherent with some one of the channels. Theoretically, the improvement in this channel would be about 1 db. However, even this improvement would be just offset by the increased false alarm number due to the multiple channels, so that the over-all system improvement would be nil. It seems that there is no way to increase system sensitivity to moving targets by signal injection.

There is some possibility of increasing sensitivity for stationary targets by coherent signal injection, but it is difficult to imagine a practical situation where such a method would be of any use.

PROBABILITY OF DETECTION WITH MOVING TARGET INDICATION SYSTEMS

The analysis of the probability of detection for MTI systems is quite complicated. It depends on the type of receiver (lin-log limiting or IAGC), the type of detector, and the characteristics of the storage device used. For a nonfluctuating clutter and no scanning noise, the effect of the clutter with or without the addition of a coherent oscillator is much the same as that of the injected carrier discussed in the previous section. If a suitable detection system is used, the sensitivity may be reduced by a small amount, due to the addition of the coho, perhaps by 1 to 3 db.

The sensitivity of an MTI system for high probabilities of detection is further reduced due to the fact that the target may be moving at a speed differing from one of the so-called optimum speeds. This effect is quite complicated and is similar to that caused by a random variation of the cross section of a target with aspect. A method of quantitatively treating these problems has been developed and will be presented in detail in a future report.

If there is a fluctuation component in the clutter, due either to the movement of the clutter itself or to the scanning of the antenna, the effect will be to increase the amount of noise at the receiver input. This can be taken into account by an appropriate adjustment in the value of the noise figure of the receiver that will change R_0 by the correct amount.

TABLES OF THE DERIVATIVES OF THE ERROR FUNCTION

In order to make efficient use of Gram-Charlier series, it is necessary to have a good table of the derivatives of the error integral (the ϕ functions of Eq. 62). No satisfactory table was in existence at the time this report was written. Typical of the available tables⁽³⁾ were

Fry⁽⁵⁾ $n = 1(1)6, x = 0(.1)4$ 5 decimals

Jorgensen $n = 1(1)6, x = 0(.01)4$ 7 decimals

and an unpublished table of the W.P.A., giving

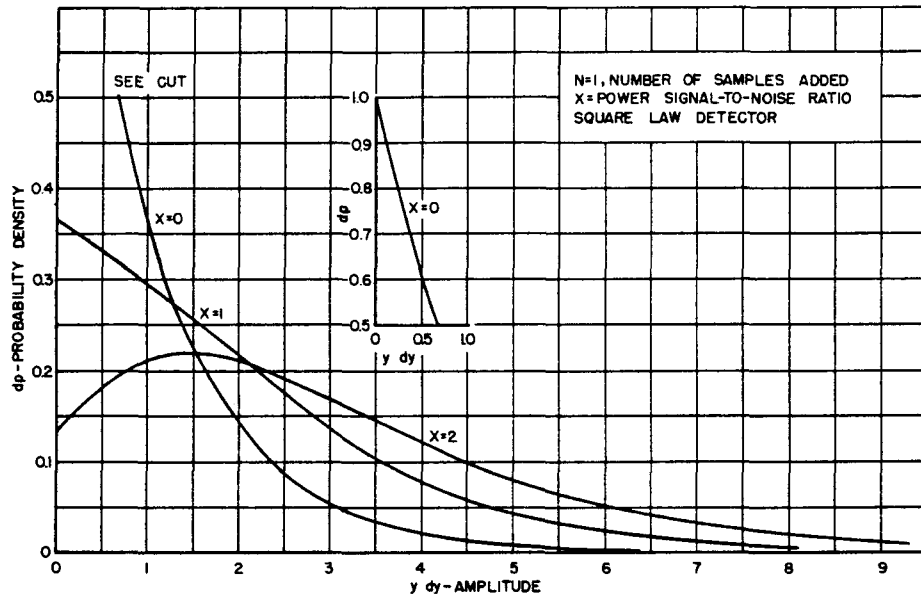
$n = 1(1)14, x = 0(.1)8.4$ 20 decimals

RAND therefore decided to calculate a suitable table with the aid of its IBM equipment. This has resulted in a table of Hermite polynomials, as well as in the derivatives of the error integral, giving

$n = 1(1)10, x = 0(.01)12.0$ 6 significant figures

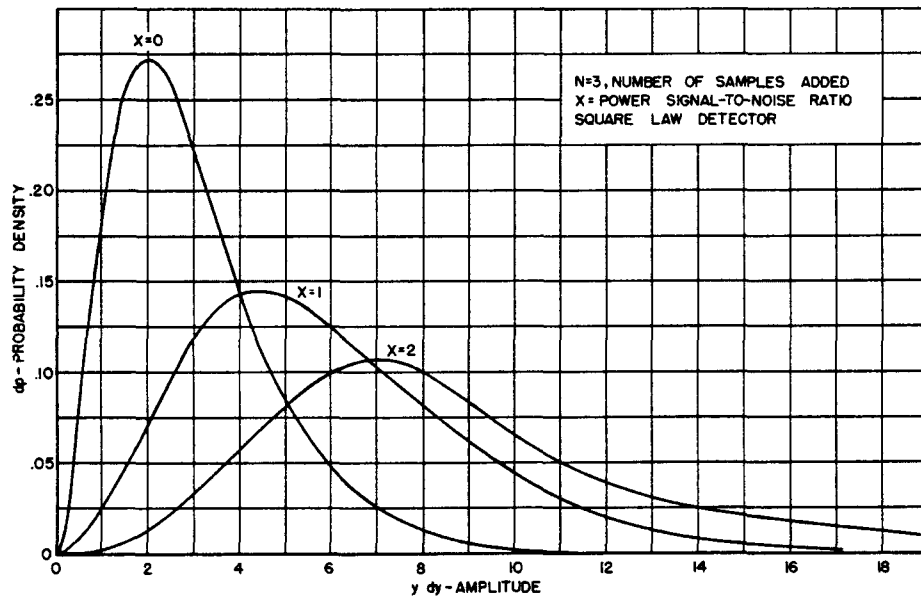
A limited number of these tables are available at the present time. (RAND Document D-350, *A Table of Hermite Polynomials and the Derivatives of the Error Function.*)





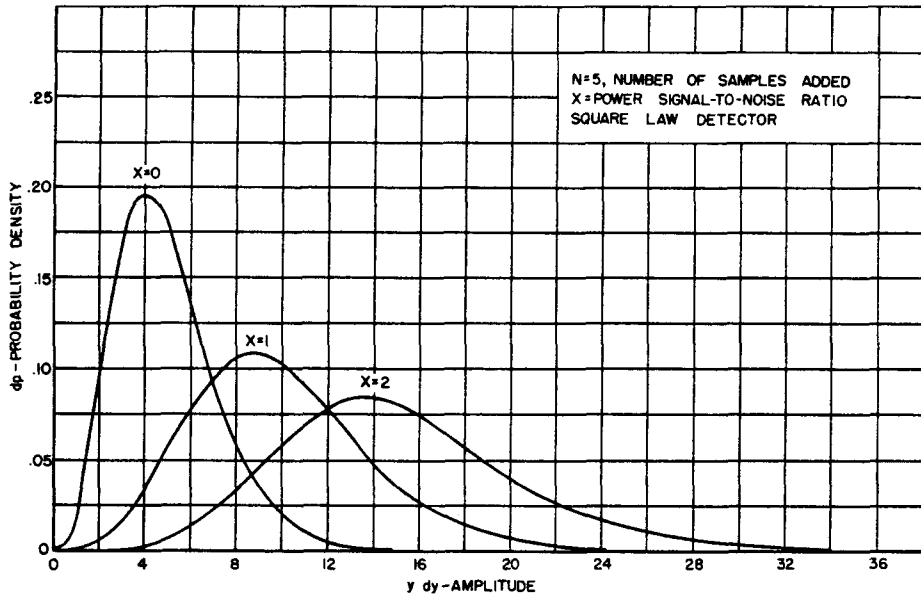
PROBABILITY DENSITY FUNCTIONS FOR ENVELOPE OF SINE WAVE PLUS NOISE

FIG. 1



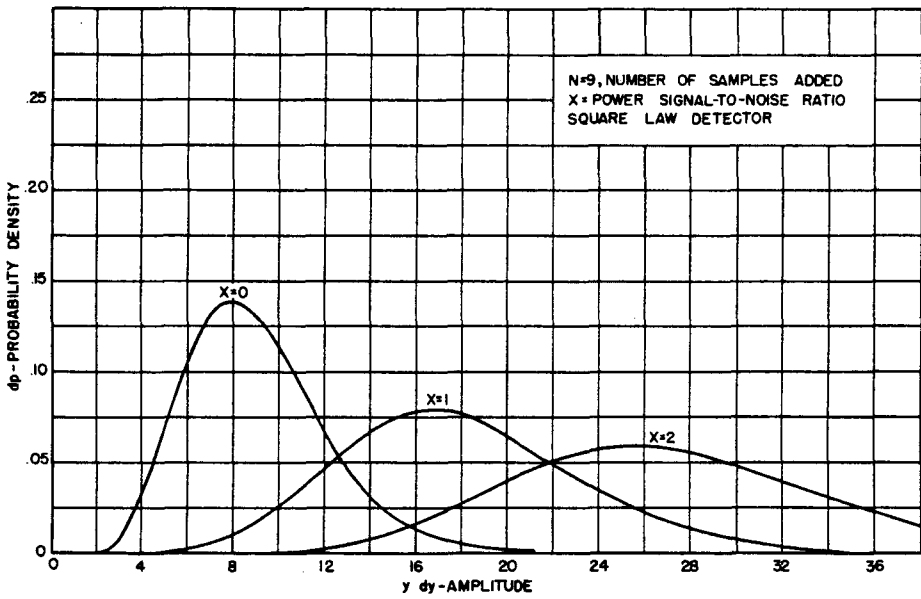
PROBABILITY DENSITY FUNCTIONS FOR ENVELOPE OF SINE WAVE PLUS NOISE

FIG. 2



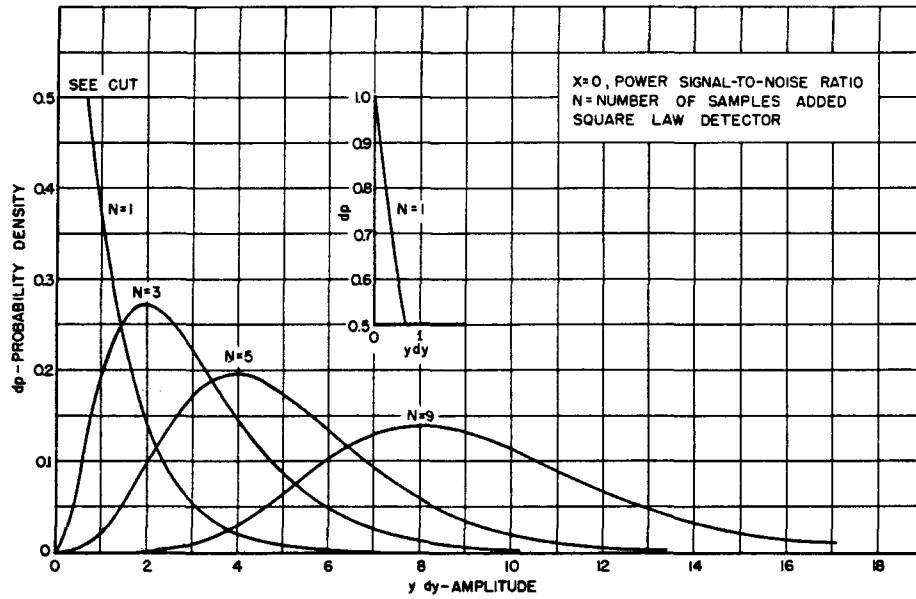
PROBABILITY DENSITY FUNCTIONS FOR ENVELOPE OF SINE WAVE PLUS NOISE

FIG. 3



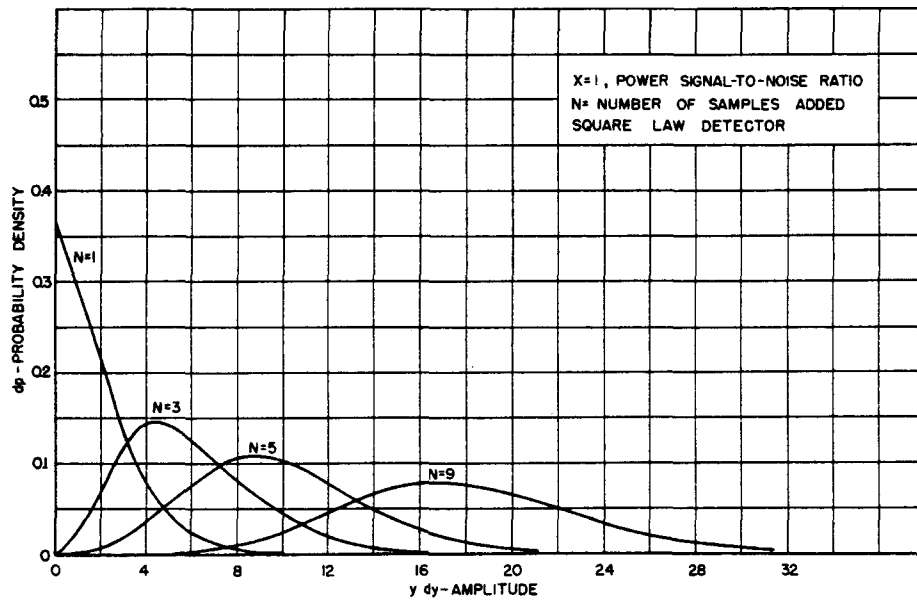
PROBABILITY DENSITY FUNCTIONS FOR ENVELOPE OF SINE WAVE PLUS NOISE

FIG. 4



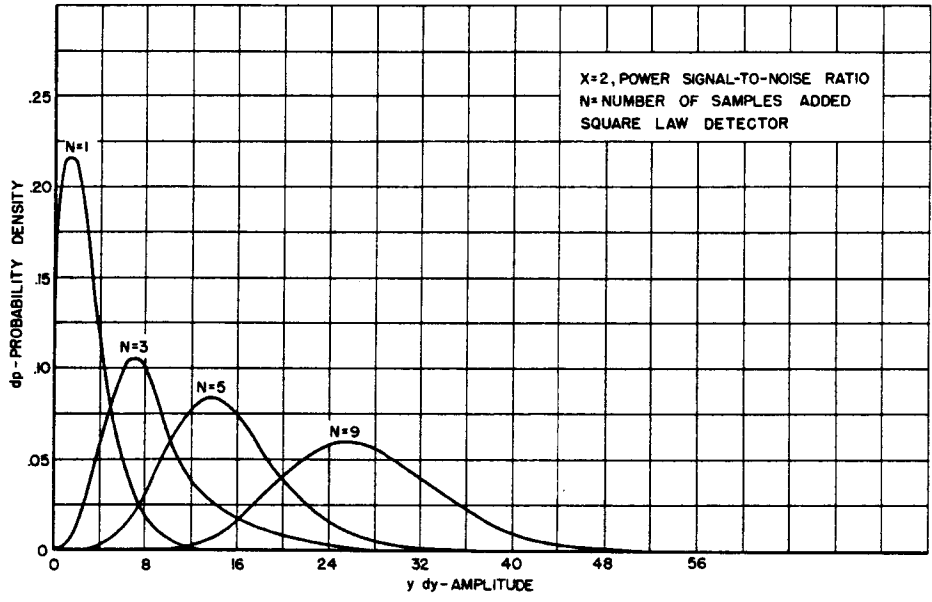
PROBABILITY DENSITY FUNCTIONS FOR ENVELOPE OF SINE WAVE PLUS NOISE

FIG 5

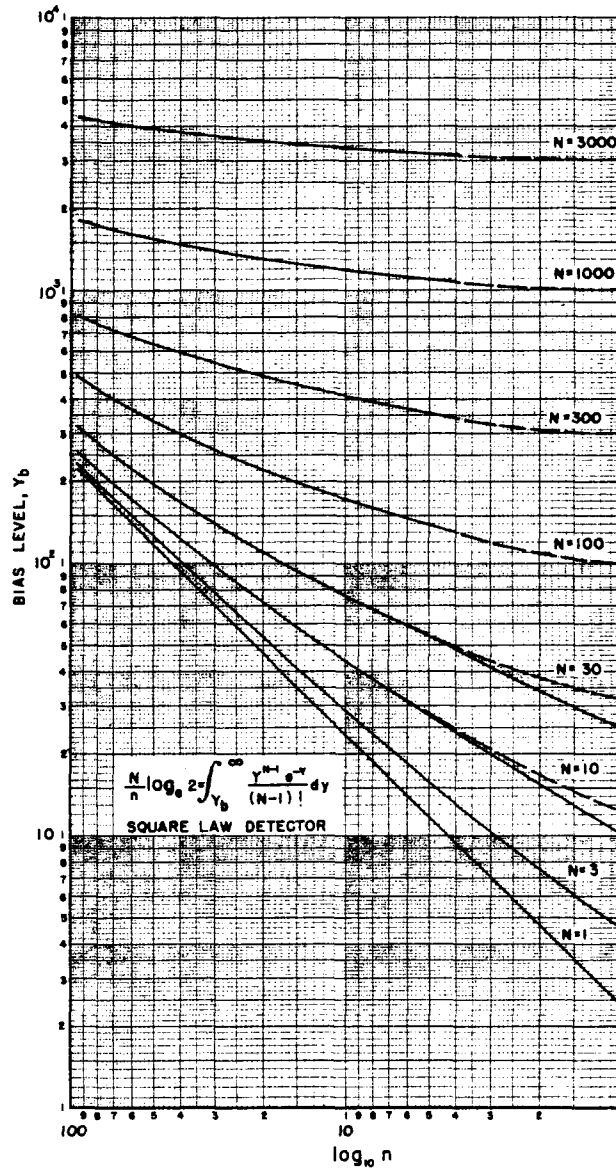


PROBABILITY DENSITY FUNCTIONS FOR ENVELOPE OF SINE WAVE PLUS NOISE

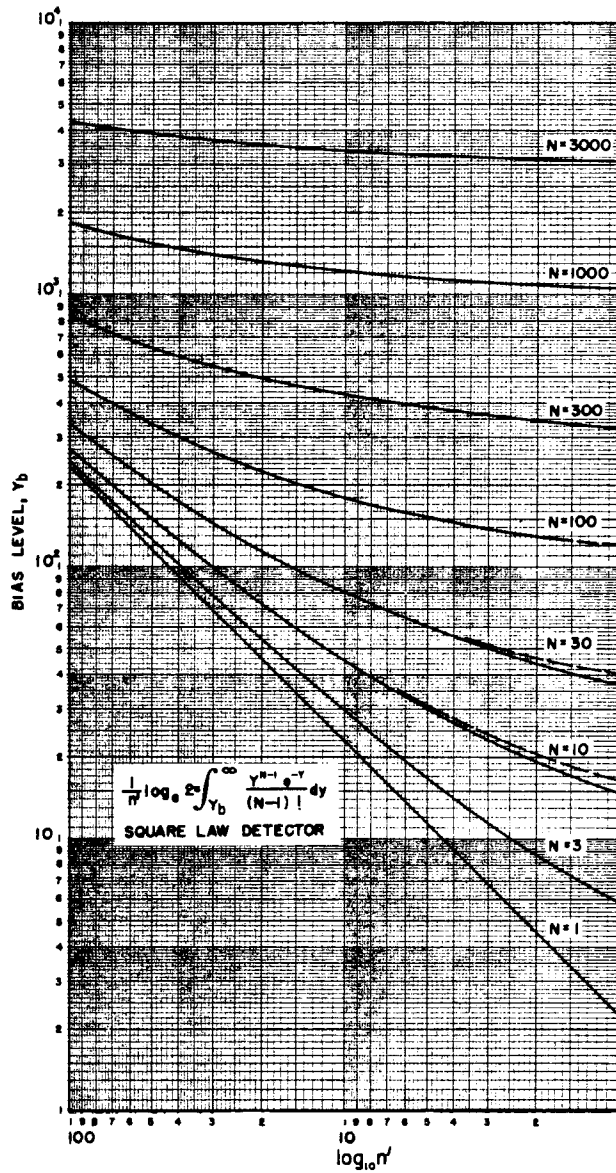
FIG 6



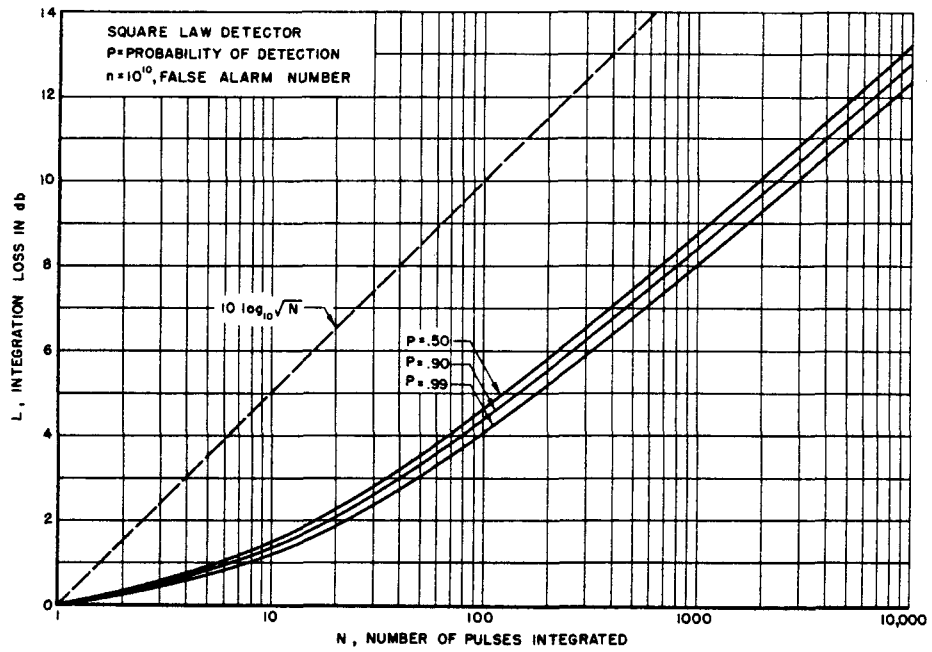
PROBABILITY DENSITY FUNCTIONS FOR ENVELOPE OF SINE WAVE PLUS NOISE
 FIG. 7



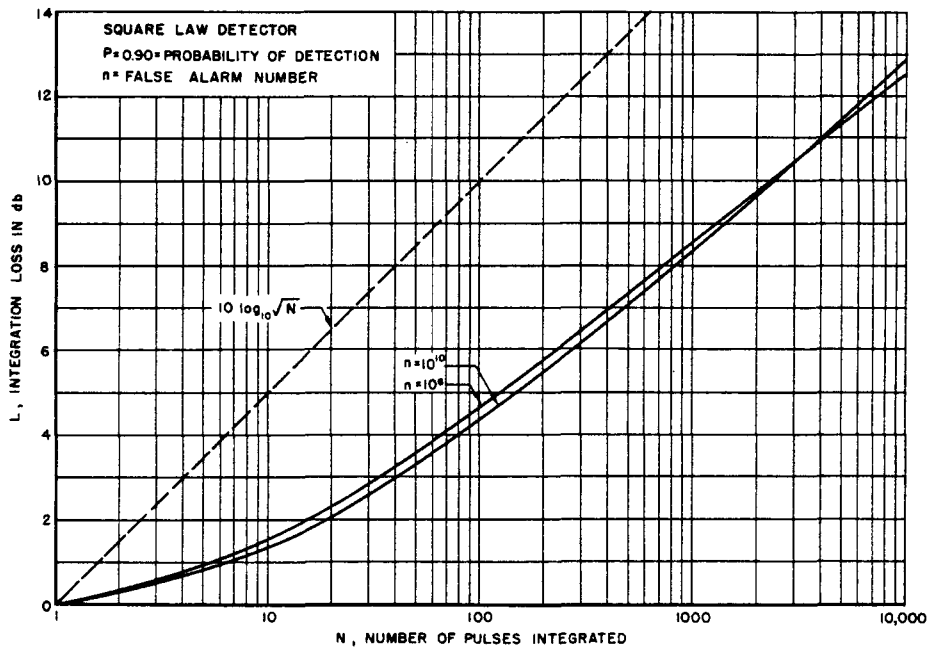
BIAS LEVEL AS A FUNCTION OF NUMBER OF PULSES INTEGRATED AND FALSE ALARM NUMBER
FIG. 8



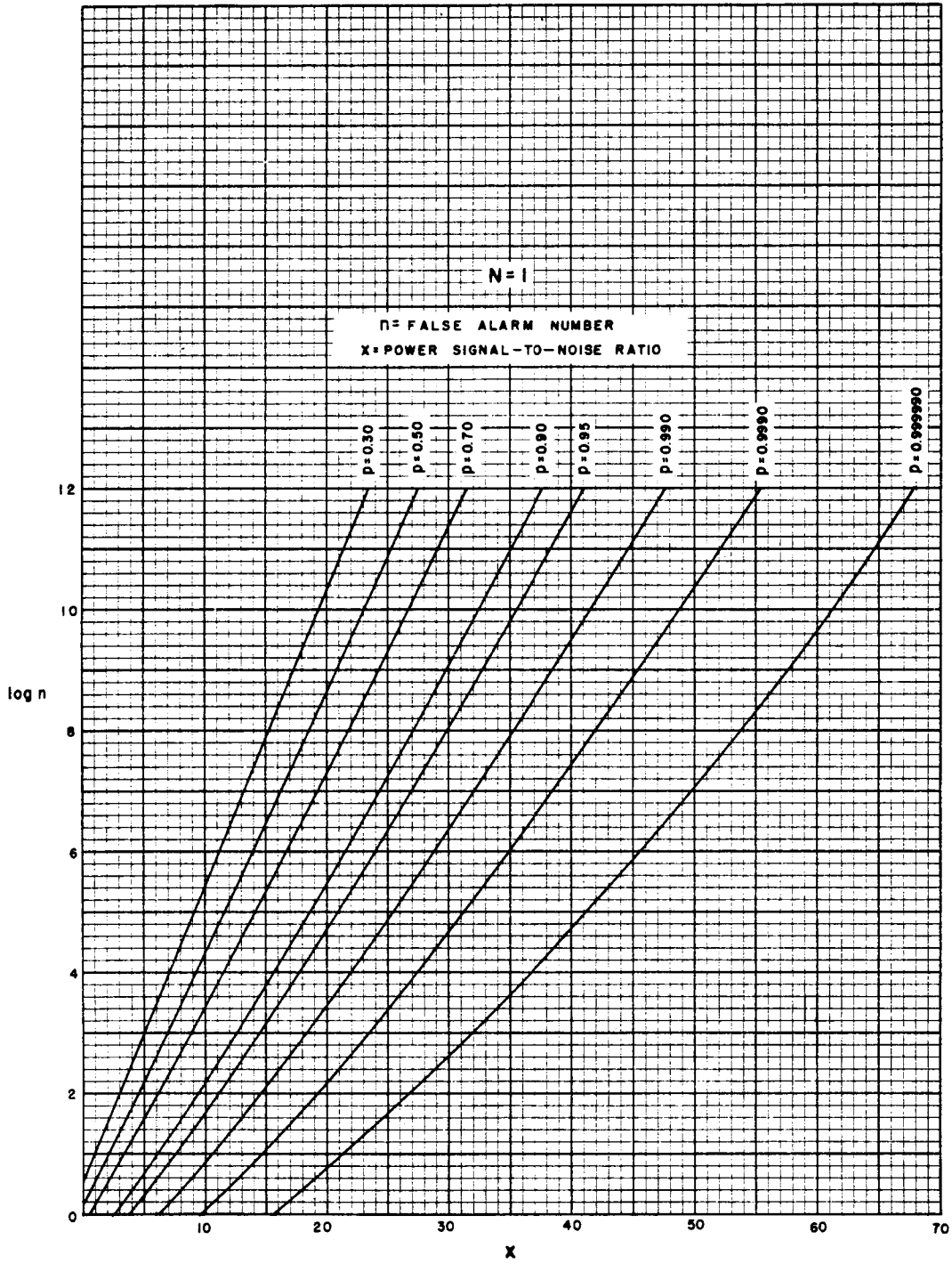
BIAS LEVEL AS A FUNCTION OF NUMBER OF PULSES INTEGRATED AND FALSE ALARM NUMBER
 FIG. 9



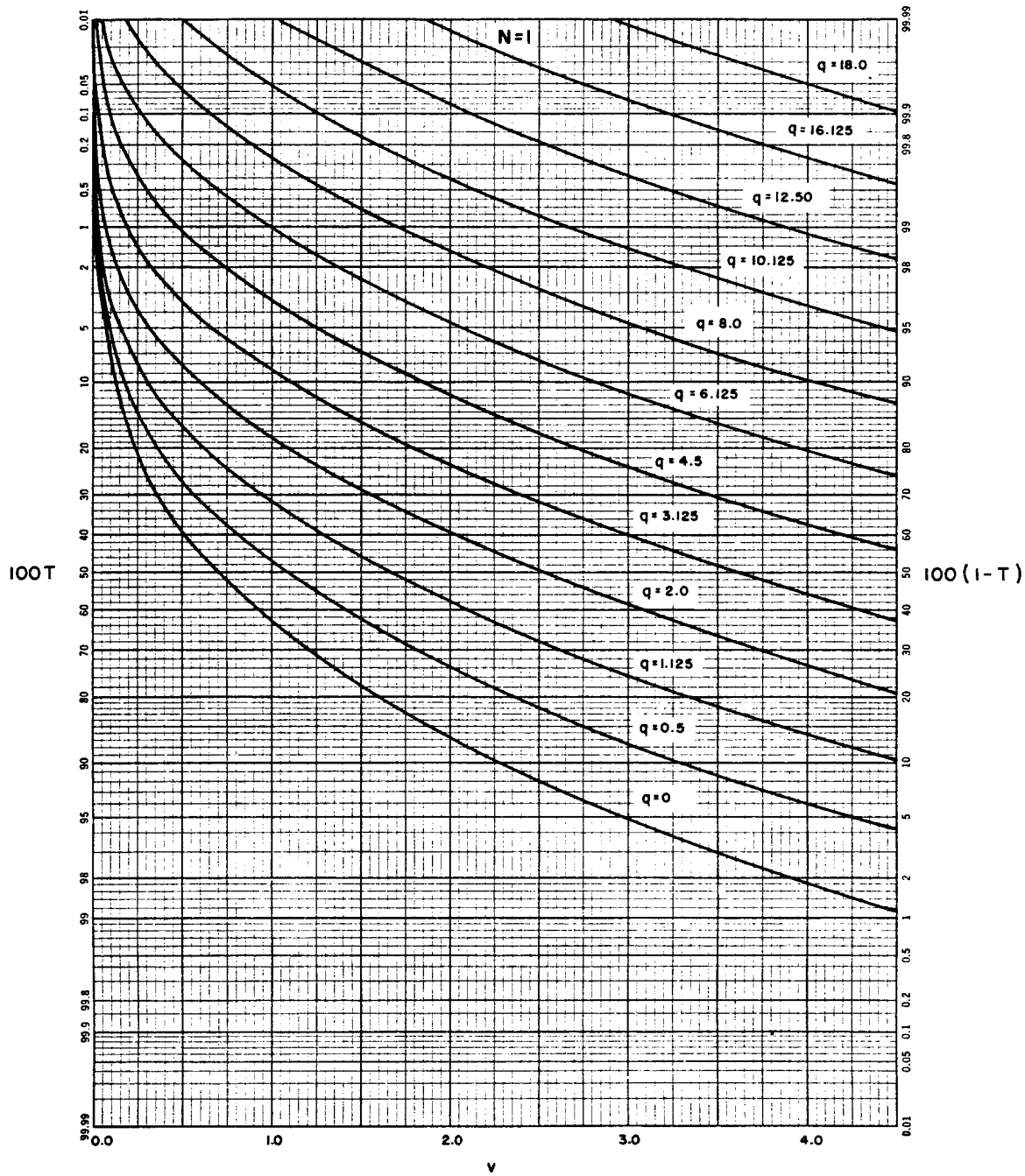
INTEGRATION LOSS, NON-COHERENT vs COHERENT
FIG.10



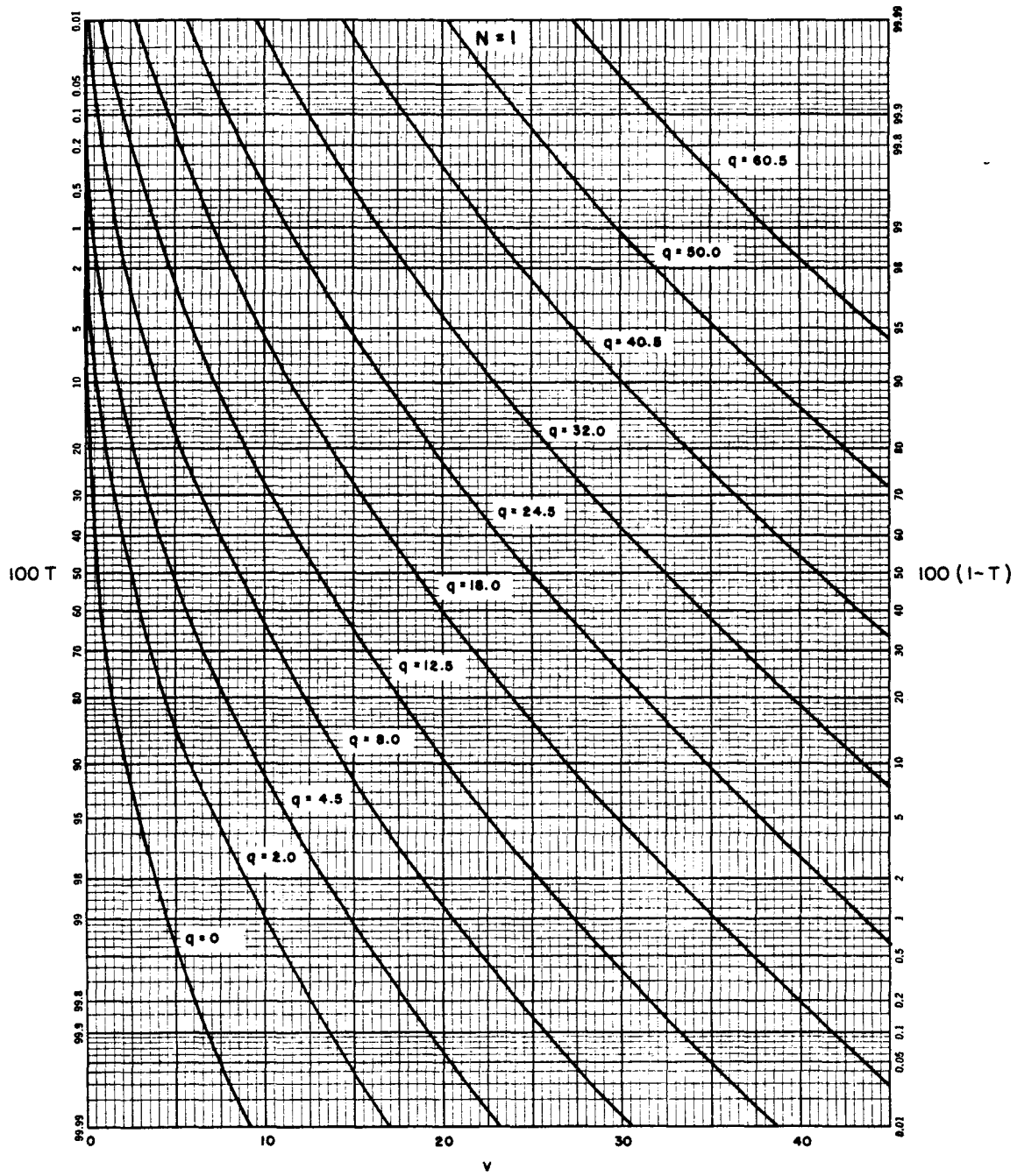
INTEGRATION LOSS, NON-COHERENT vs COHERENT
FIG.11



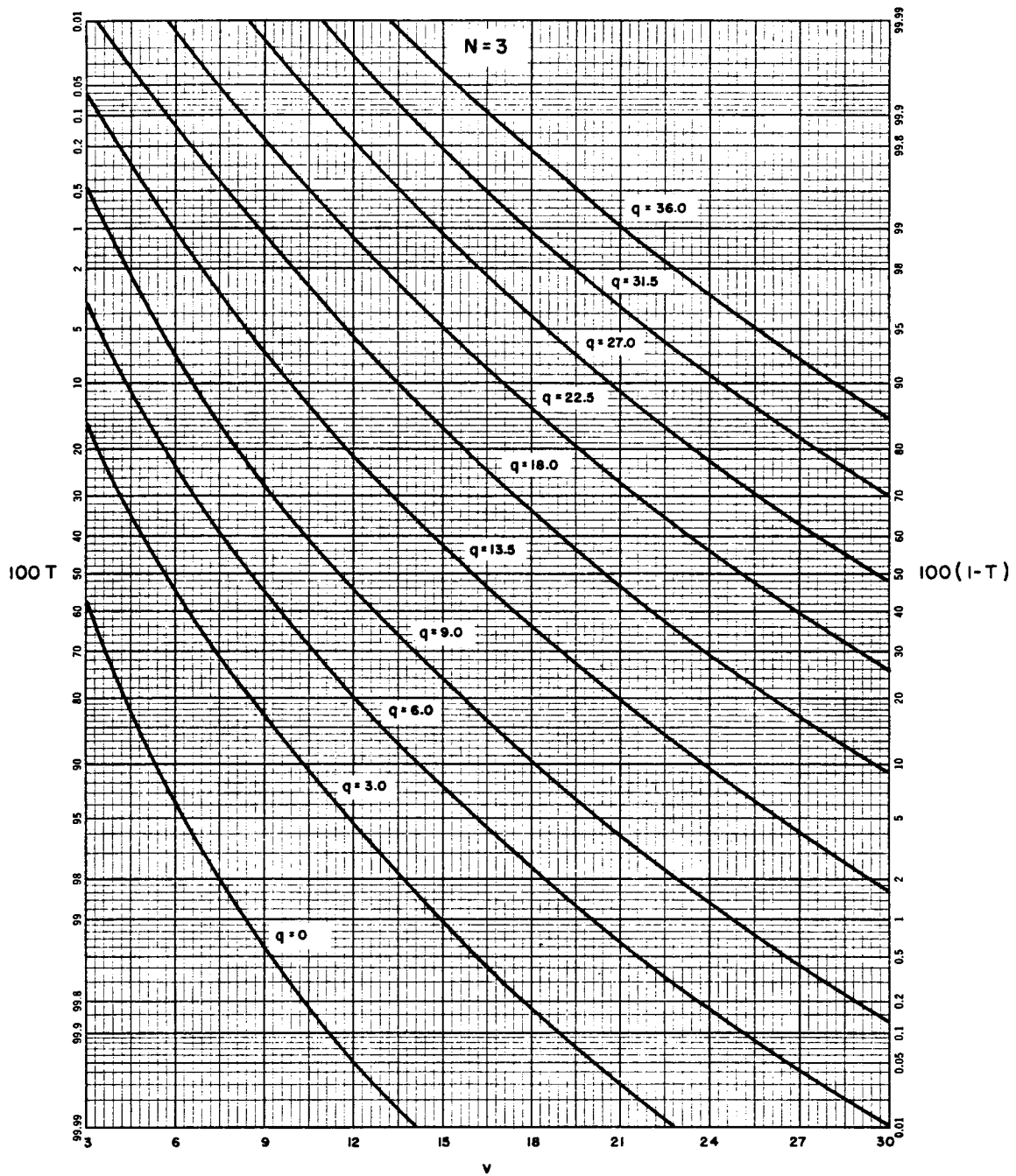
PROBABILITY OF DETECTION WITH NO INTEGRATION
FIG.12



THE INCOMPLETE TORONTO FUNCTION $T_{v\sqrt{q}}$ ($2N-1, N-1, \sqrt{q}$)
 FIG.13

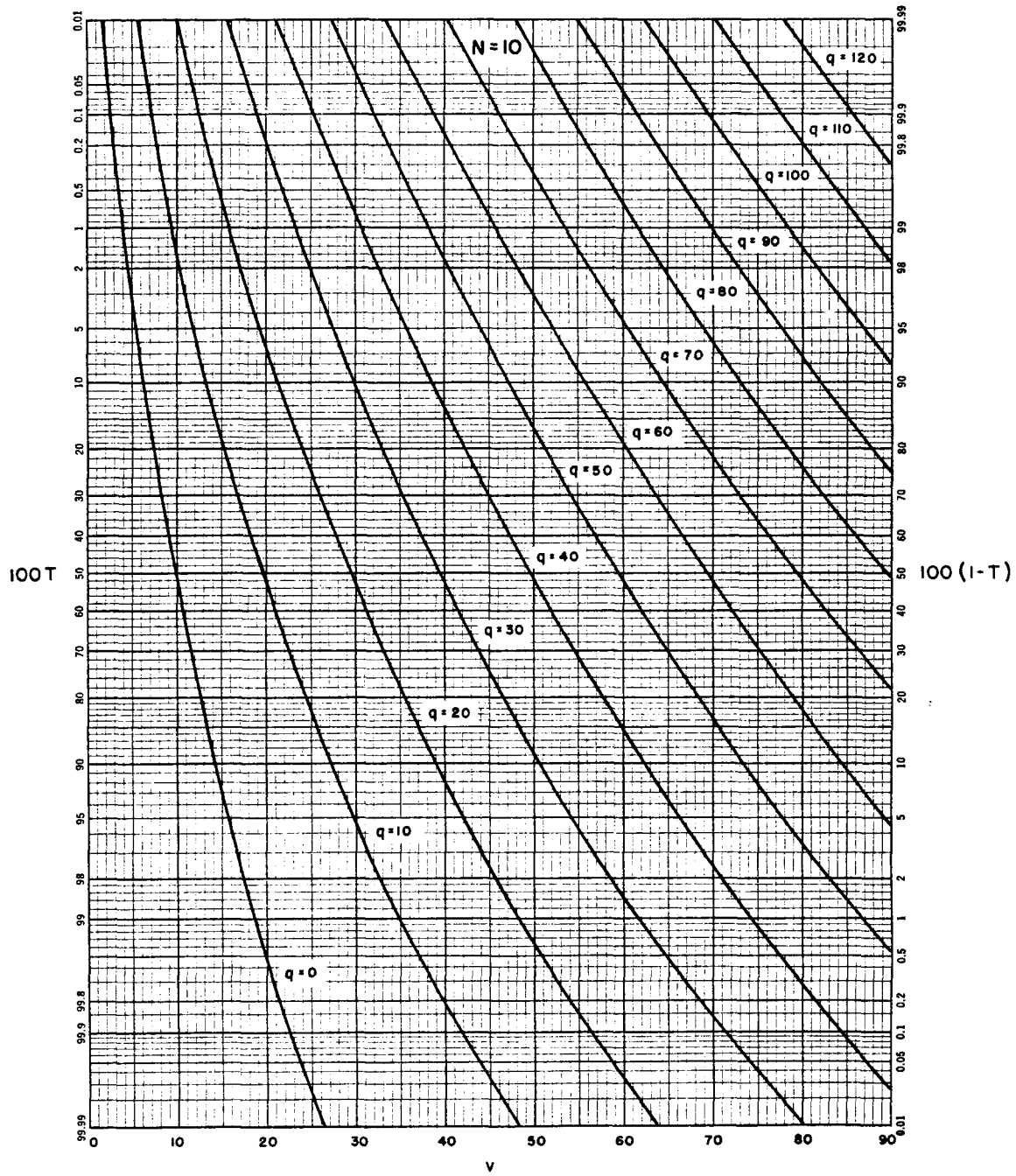


THE INCOMPLETE TORONTO FUNCTION $T_{\sqrt{q}}(2N-1, N-1, \sqrt{q})$
 FIG.14

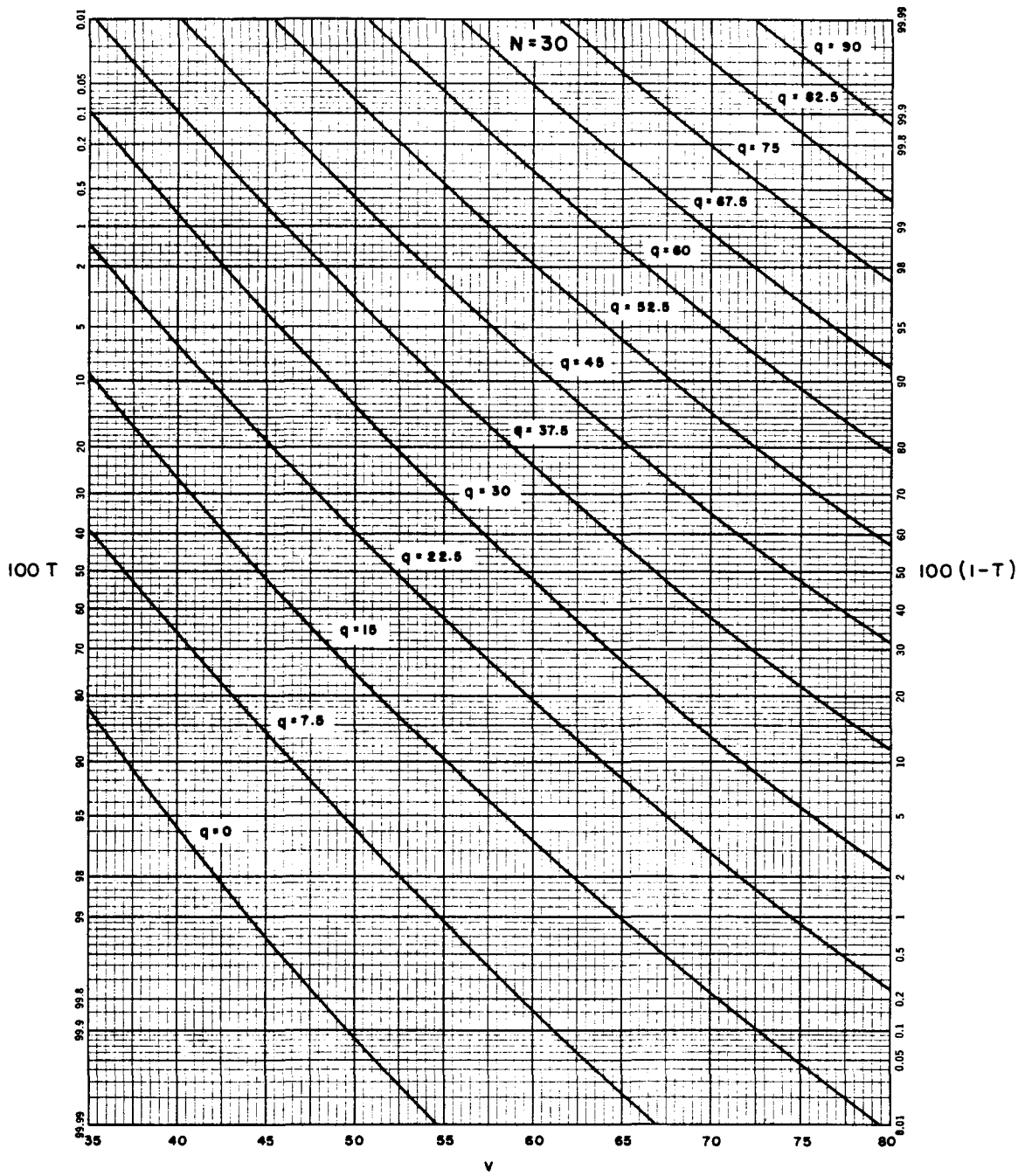


THE INCOMPLETE TORONTO FUNCTION $T_{\sqrt{q}}(2N-1, N-1, \sqrt{q})$

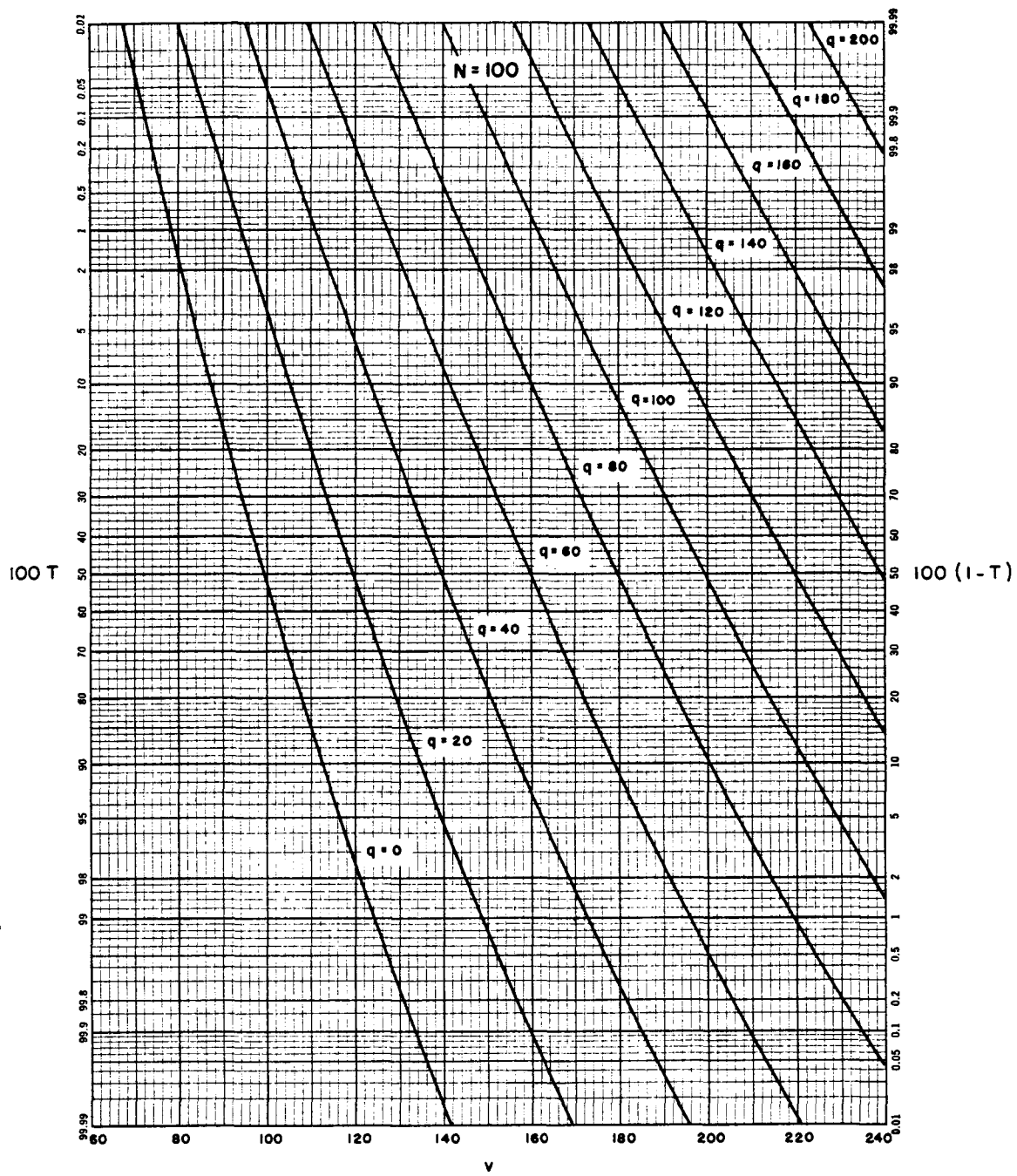
FIG. 15



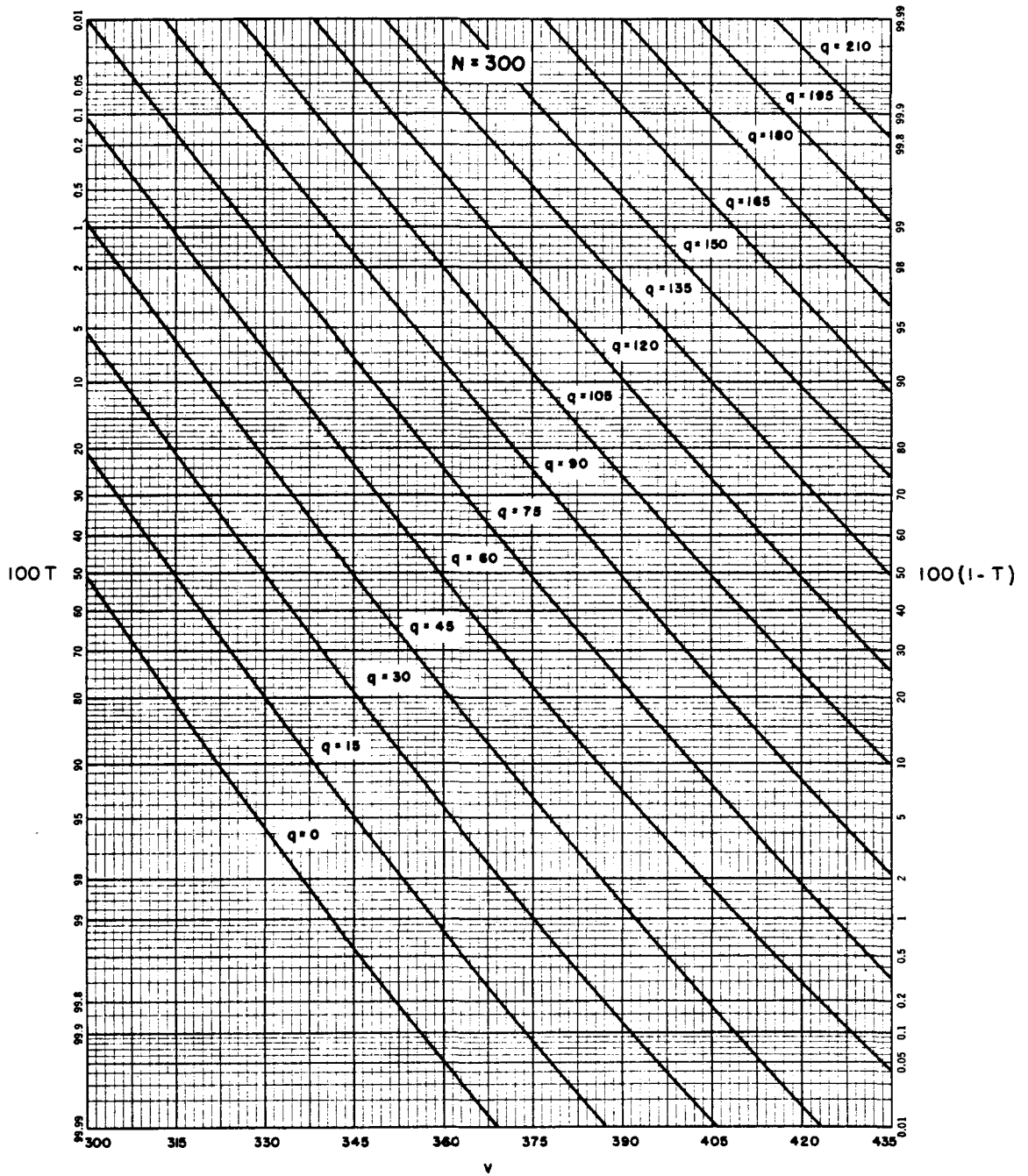
THE INCOMPLETE TORONTO FUNCTION $T_{\sqrt{v}}(2N-1, N-1, \sqrt{v})$
 FIG.16



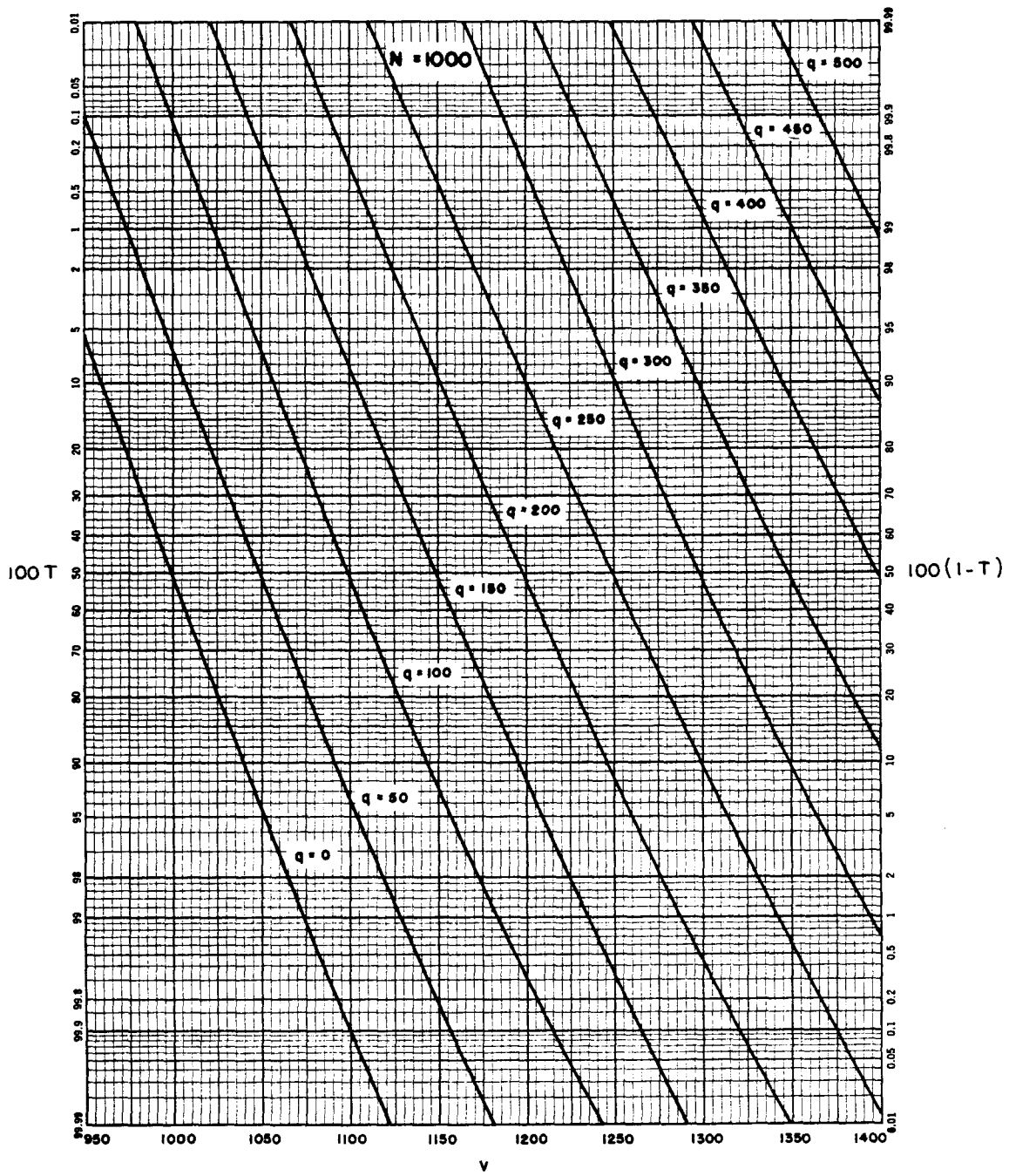
THE INCOMPLETE TORONTO FUNCTION $T_{\sqrt{v}}(2N-1, N-1, \sqrt{v})$
 FIG.17



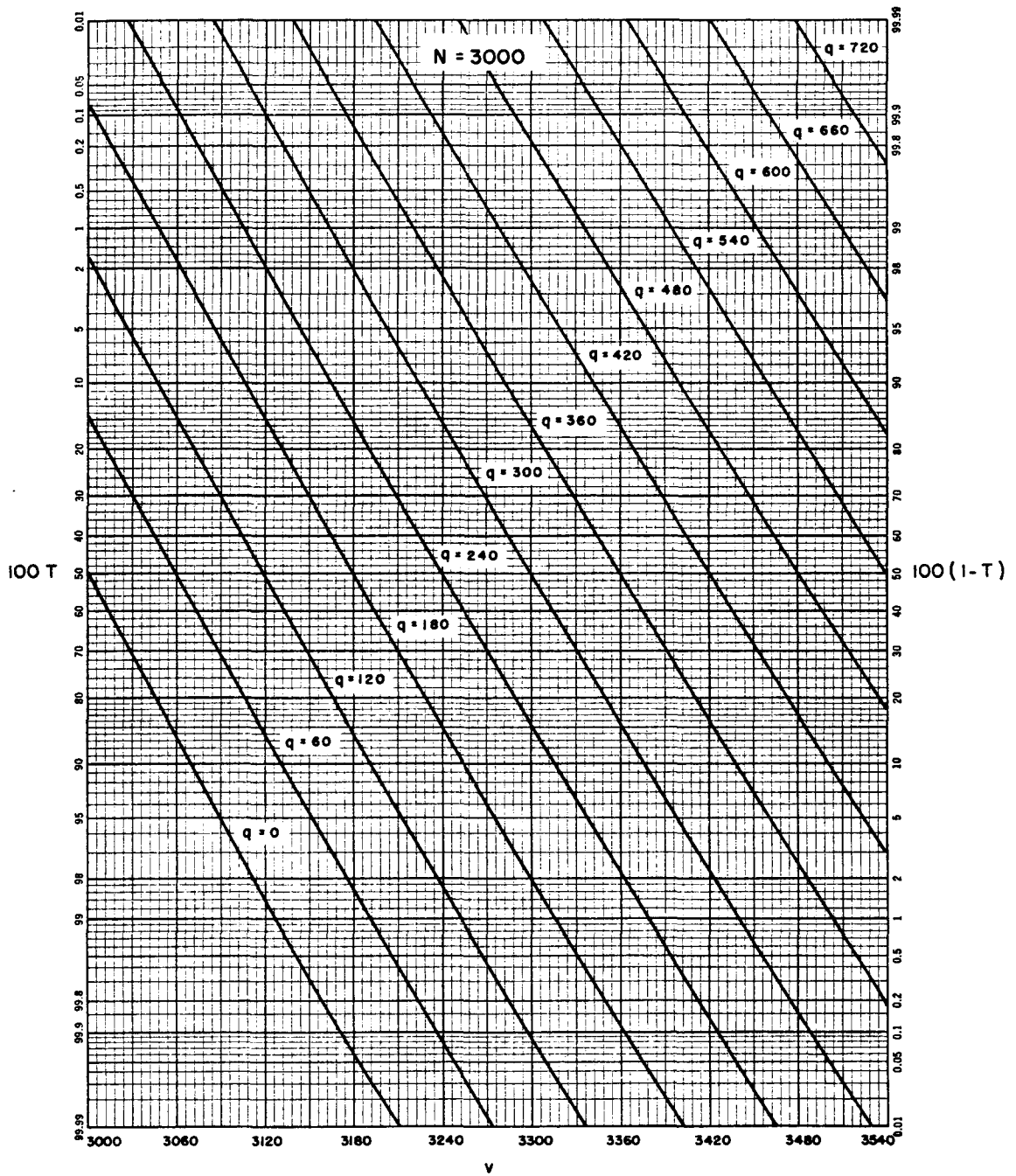
THE INCOMPLETE TORONTO FUNCTION $T_{\sqrt{v}}(2N-1, N-1, \sqrt{q})$
 FIG.18



THE INCOMPLETE TORONTO FUNCTION $T_{\sqrt{v}}$ ($2N-1, N-1, \sqrt{v}$)
 FIG.19

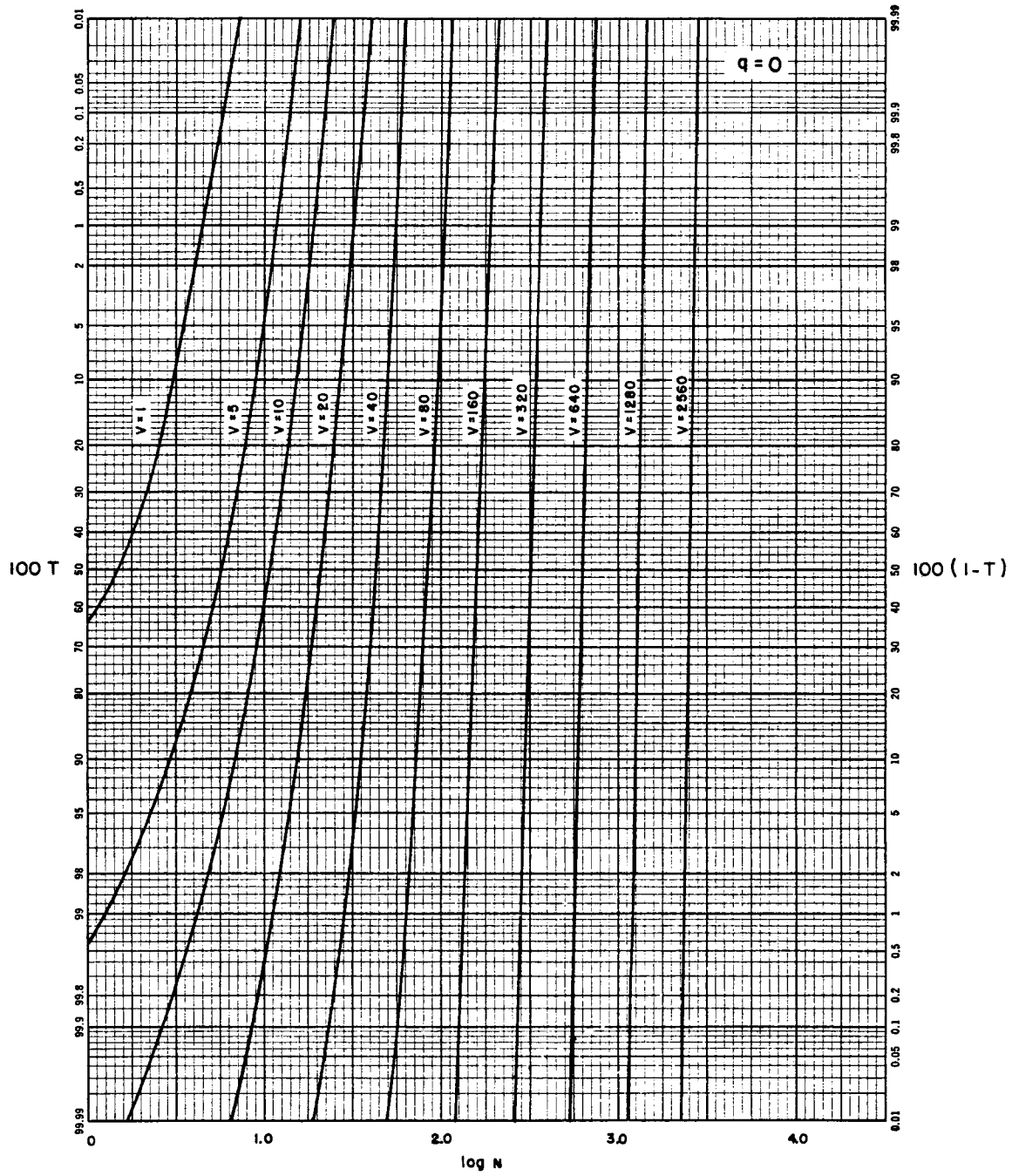


THE INCOMPLETE TORONTO FUNCTION $T_{\nu\nu} (2N-1, N-1, \sqrt{q})$
 FIG. 20

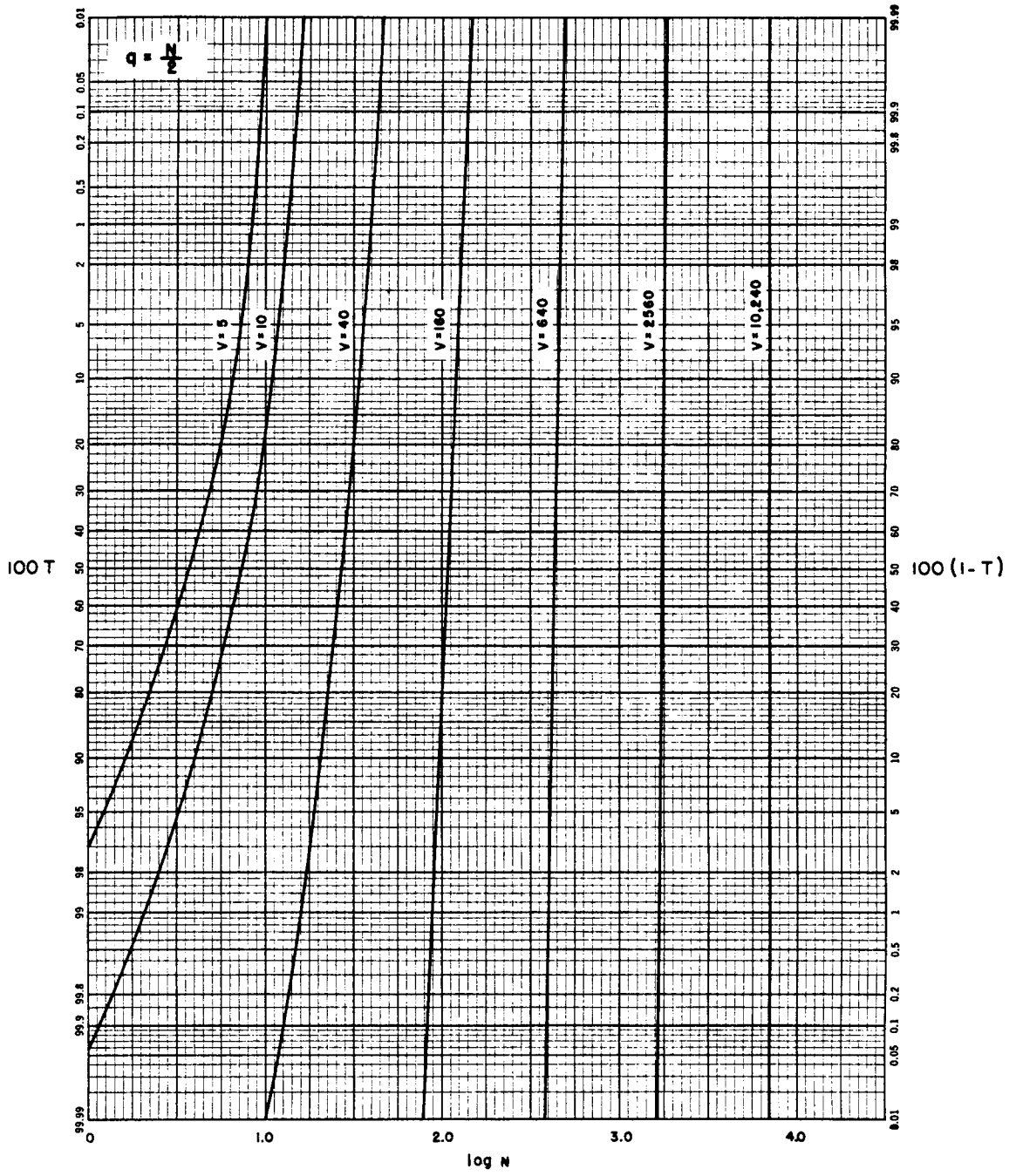


THE INCOMPLETE TORONTO FUNCTION $T_{\sqrt{v}}(2N-1, N-1, \sqrt{q})$

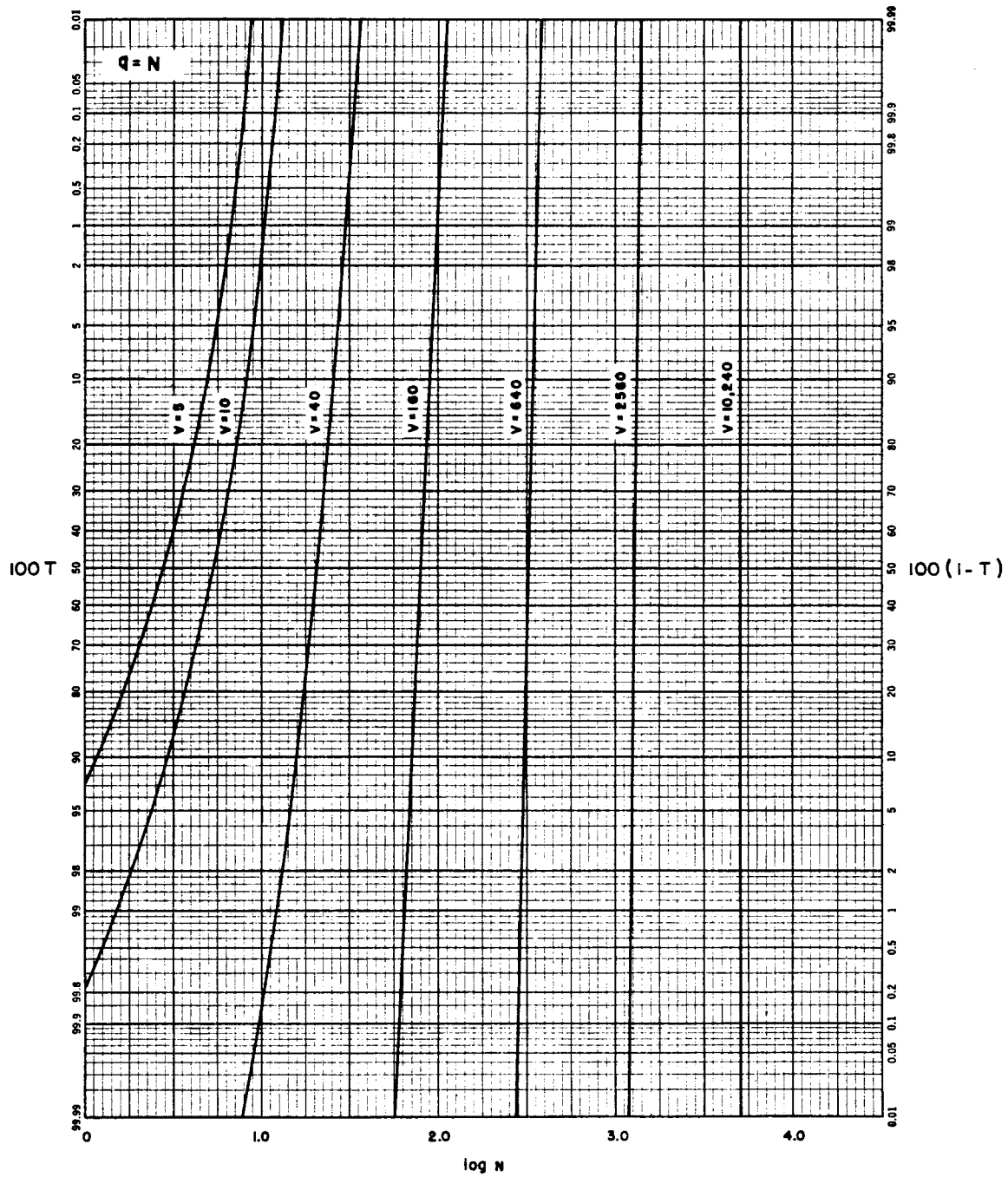
FIG. 21



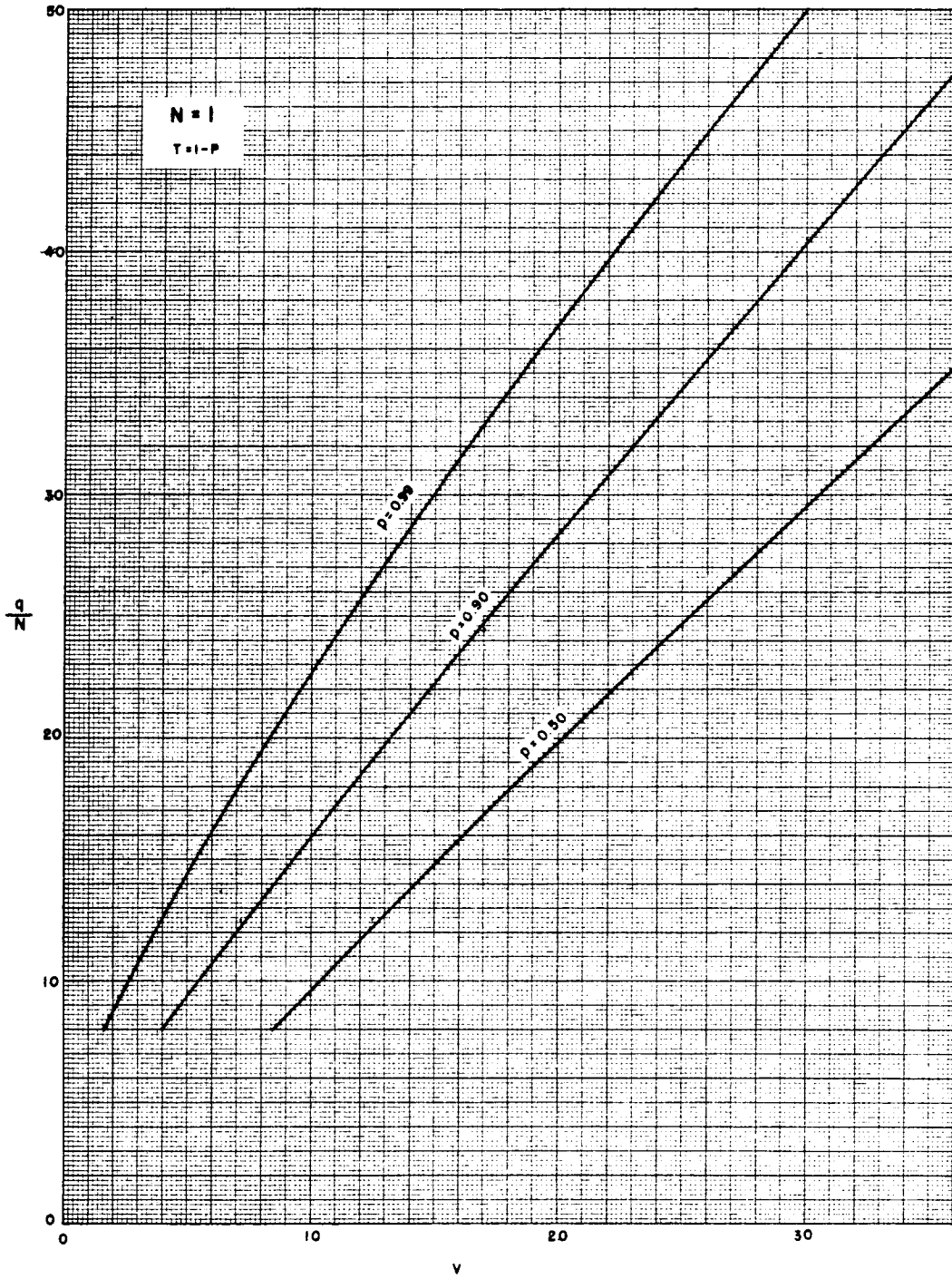
THE INCOMPLETE TORONTO FUNCTION $T_{\sqrt{q}}(2N-1, N-1, \sqrt{q})$
 FIG. 22



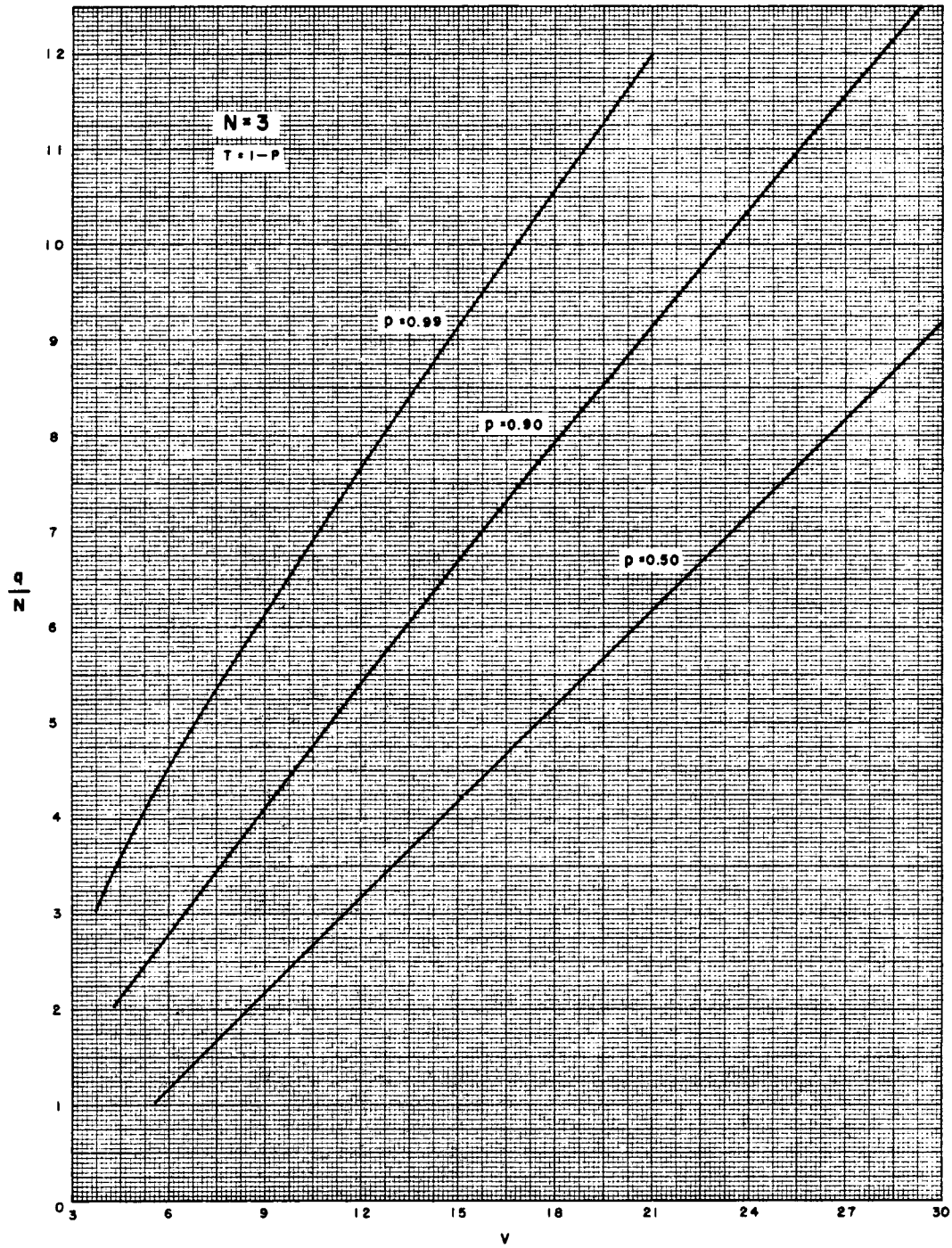
THE INCOMPLETE TORONTO FUNCTION $T_{\sqrt{q}}(2N-1, N-1, \sqrt{q})$
 FIG. 23



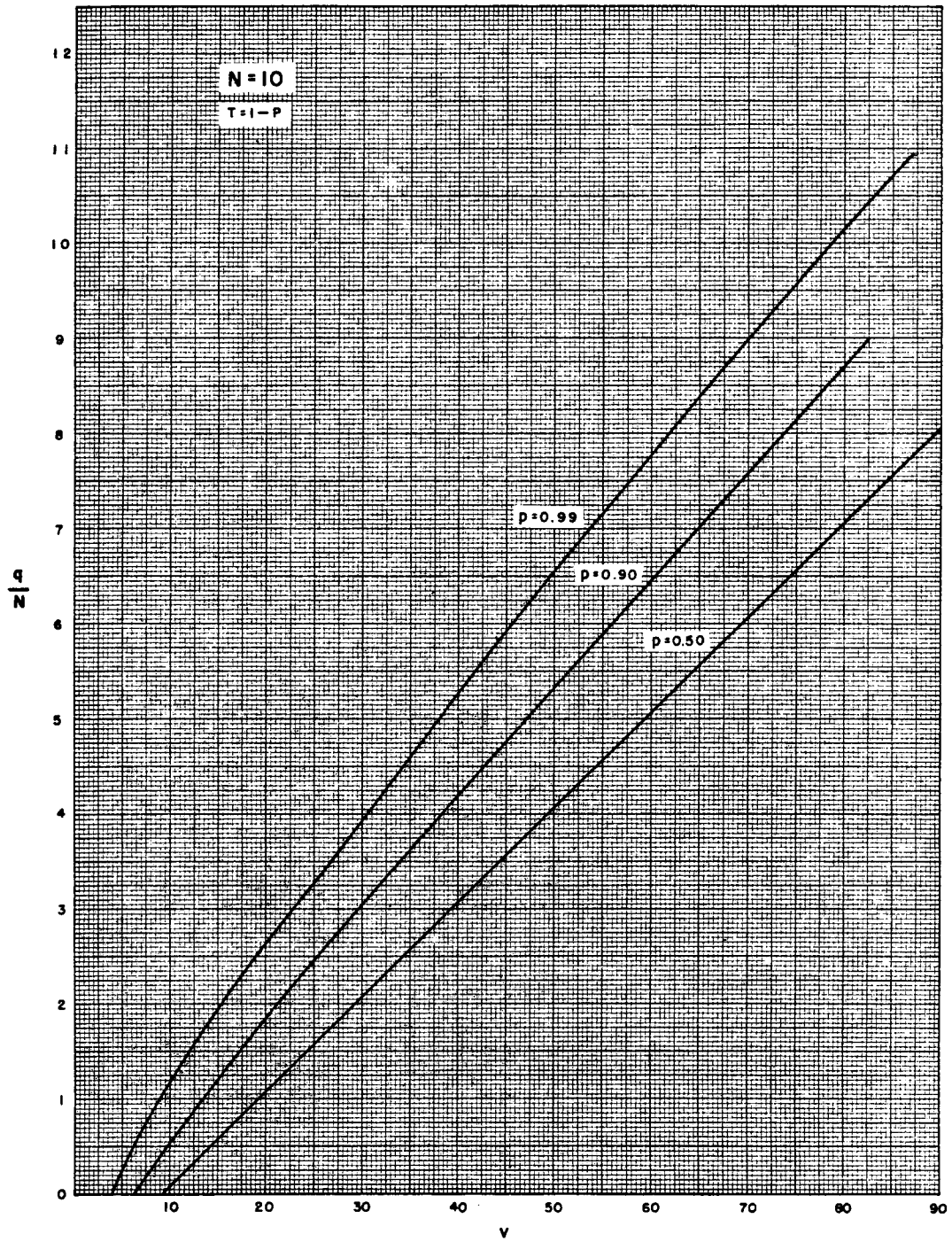
THE INCOMPLETE TORONTO FUNCTION $T_{\sqrt{q}}(2N-1, N-1, \sqrt{q})$
 FIG. 24



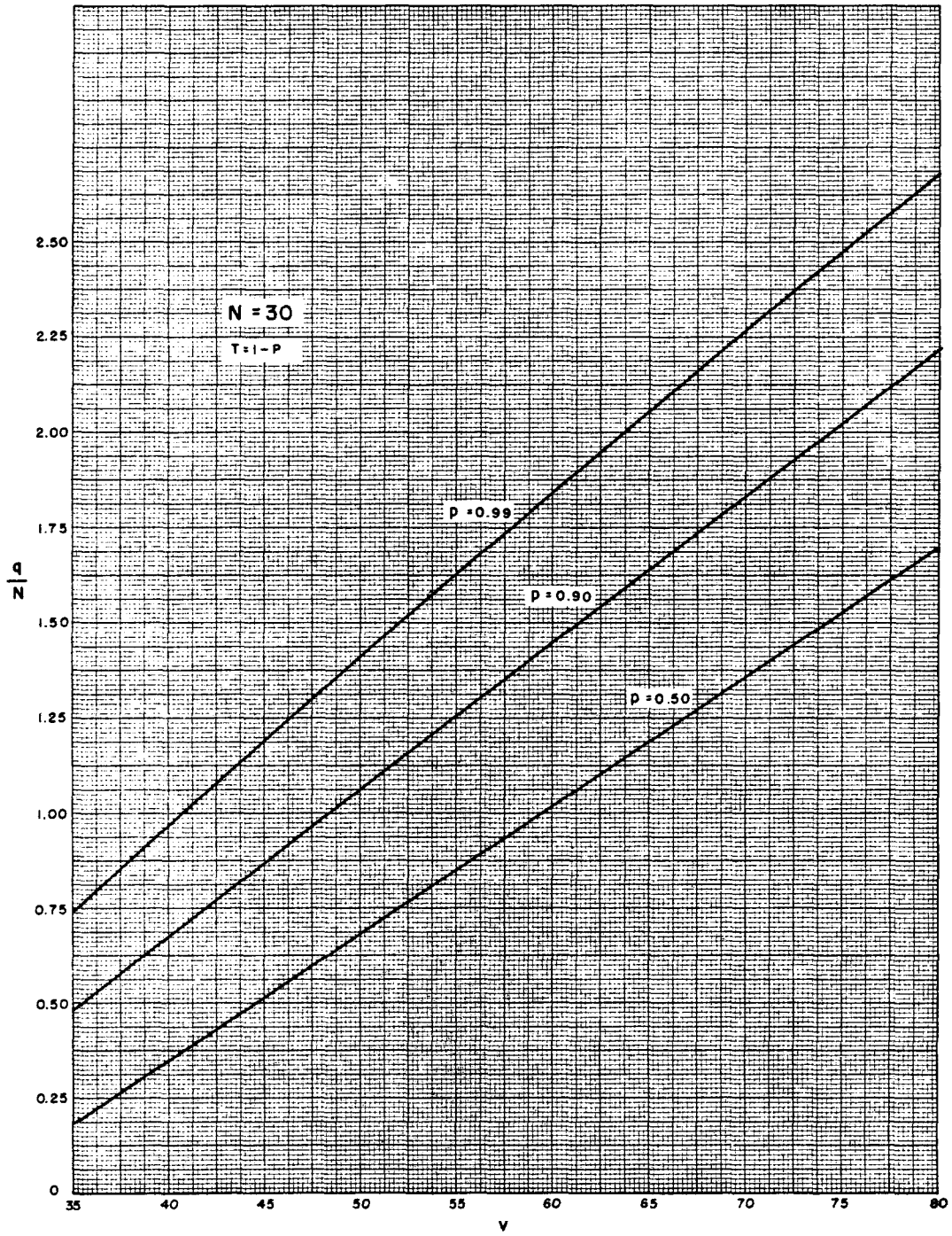
THE INCOMPLETE TORONTO FUNCTION $T_{\sqrt{q}}$ ($2N-1, N-1, \sqrt{q}$)
 FIG. 25



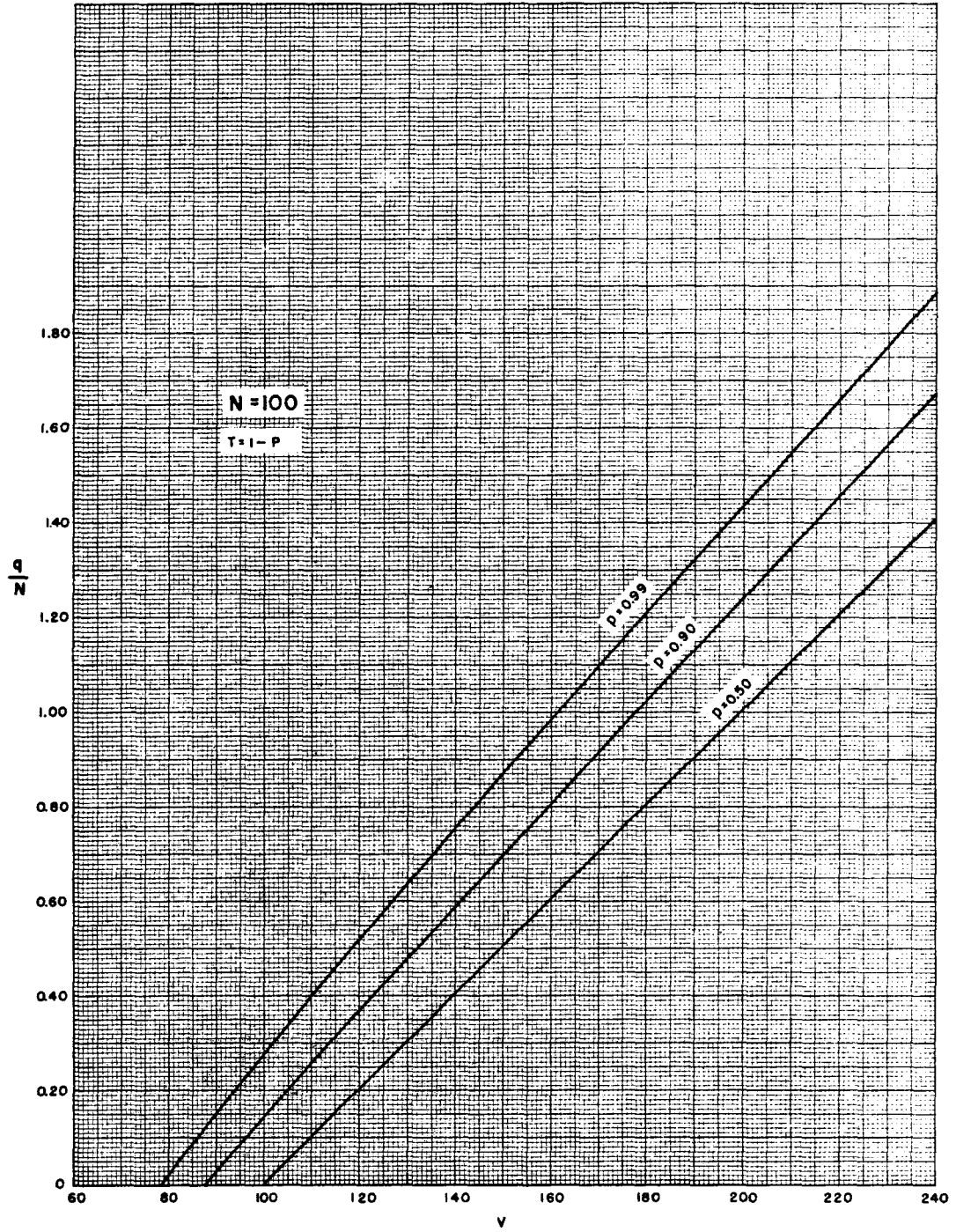
THE INCOMPLETE TORONTO FUNCTION $T_{\sqrt{v}}(2N-1, N-1, \sqrt{q})$
 FIG. 26



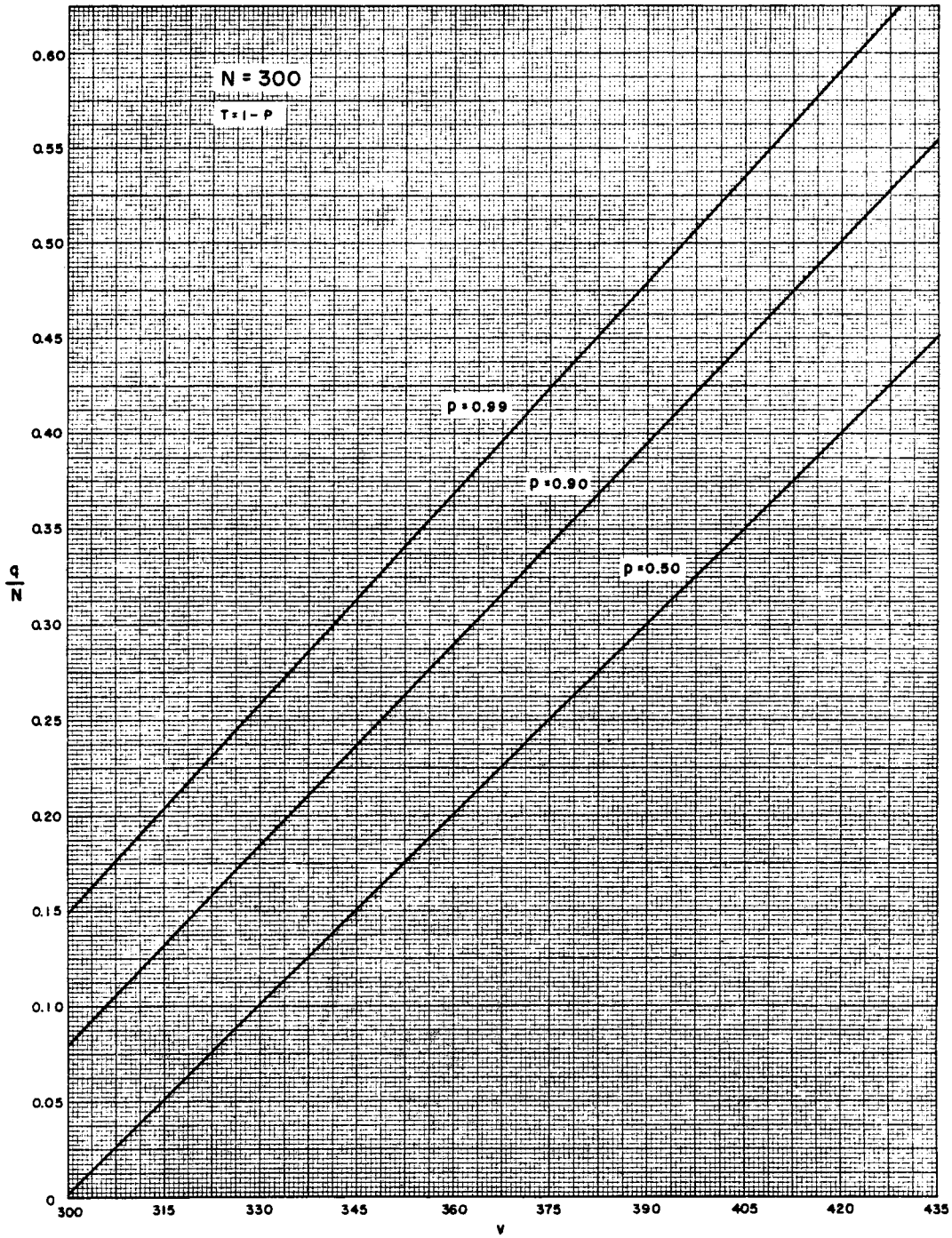
THE INCOMPLETE TORONTO FUNCTION $T_{\sqrt{q}}(2N-1, N-1, \sqrt{q})$
 FIG.27



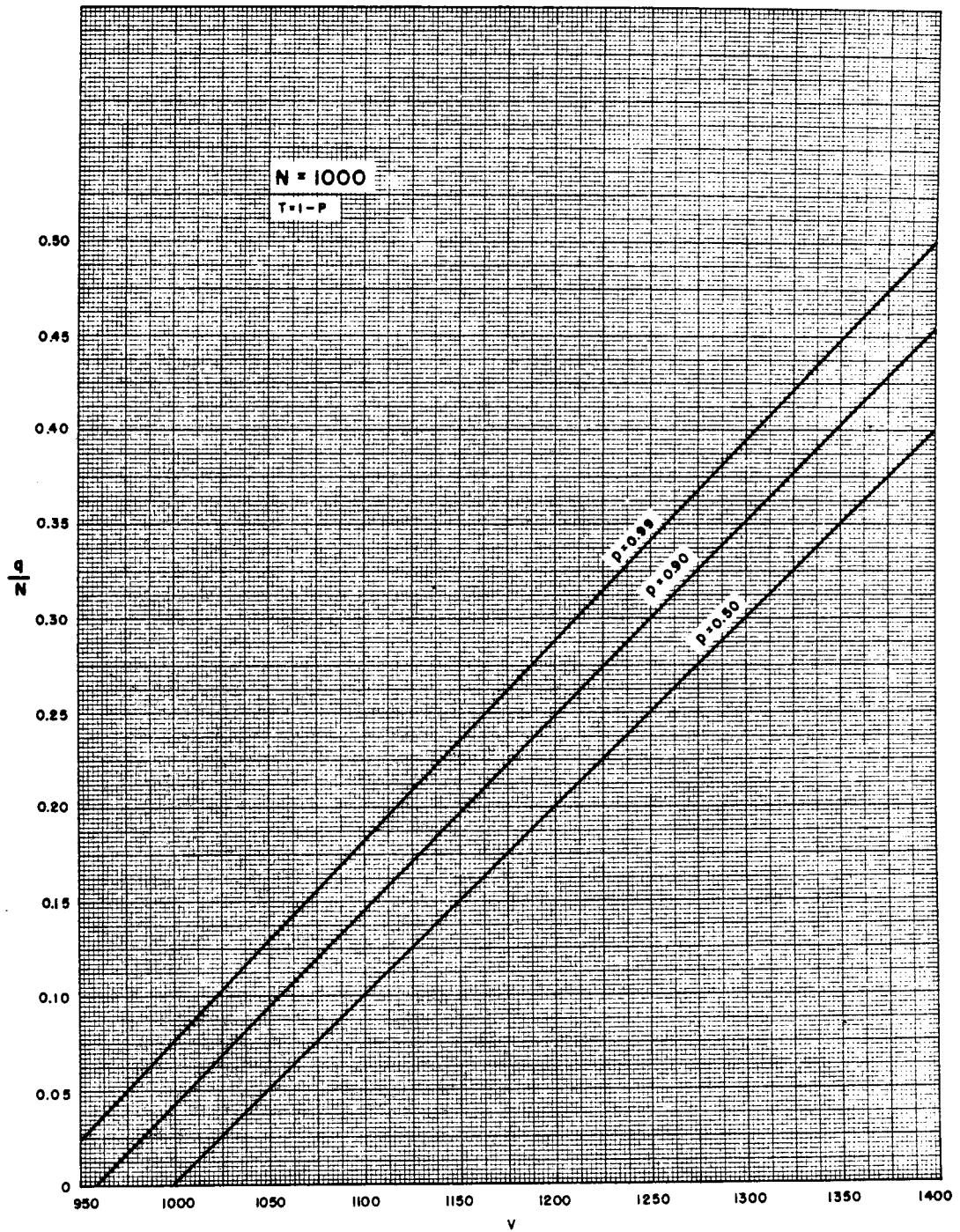
THE INCOMPLETE TORONTO FUNCTION $T_{\sqrt{q}} (2N-1, N-1, \sqrt{q})$
 FIG. 28



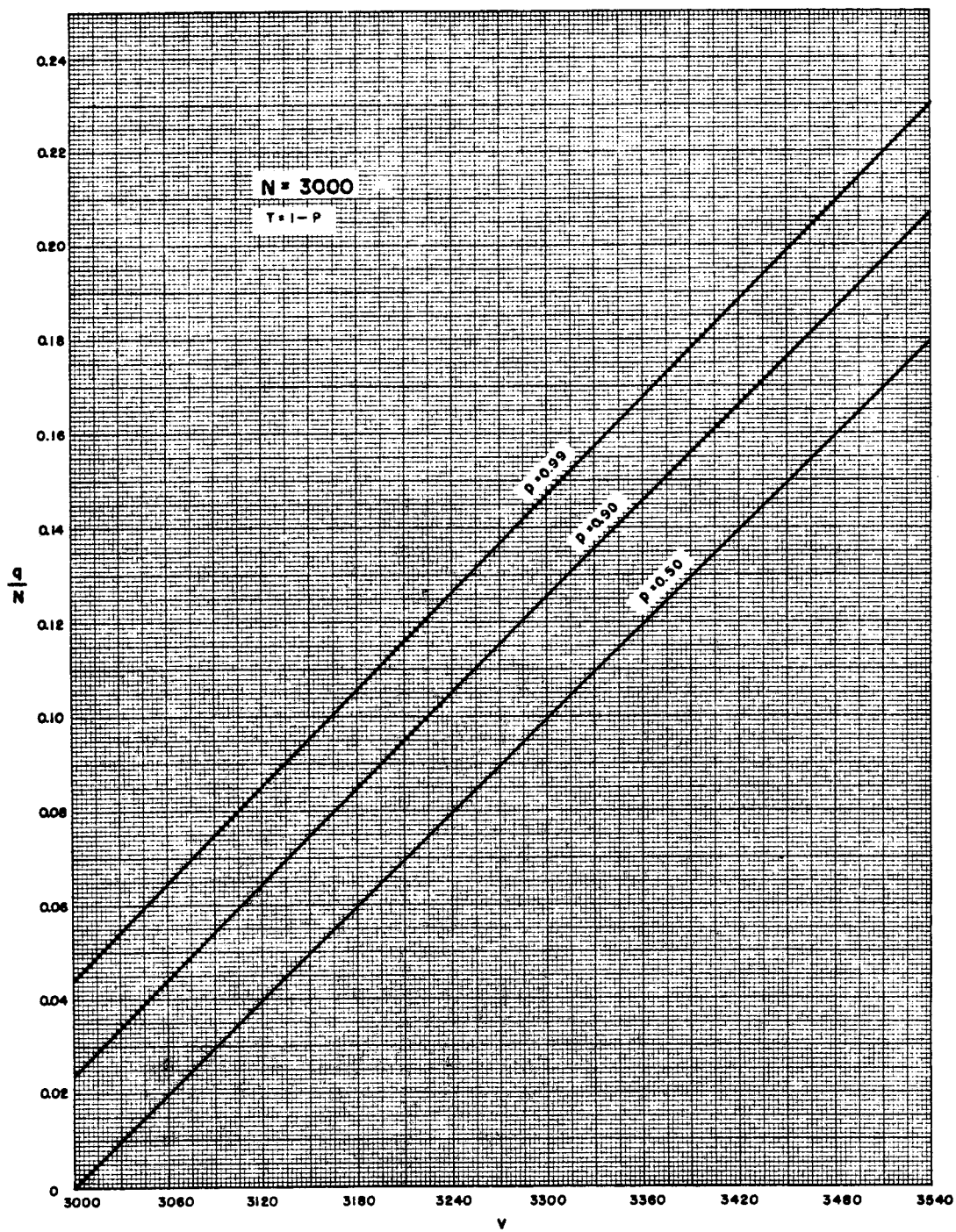
THE INCOMPLETE TORONTO FUNCTION $T_{\sqrt{v}}(2N-1, N-1, \sqrt{v})$
 FIG. 29



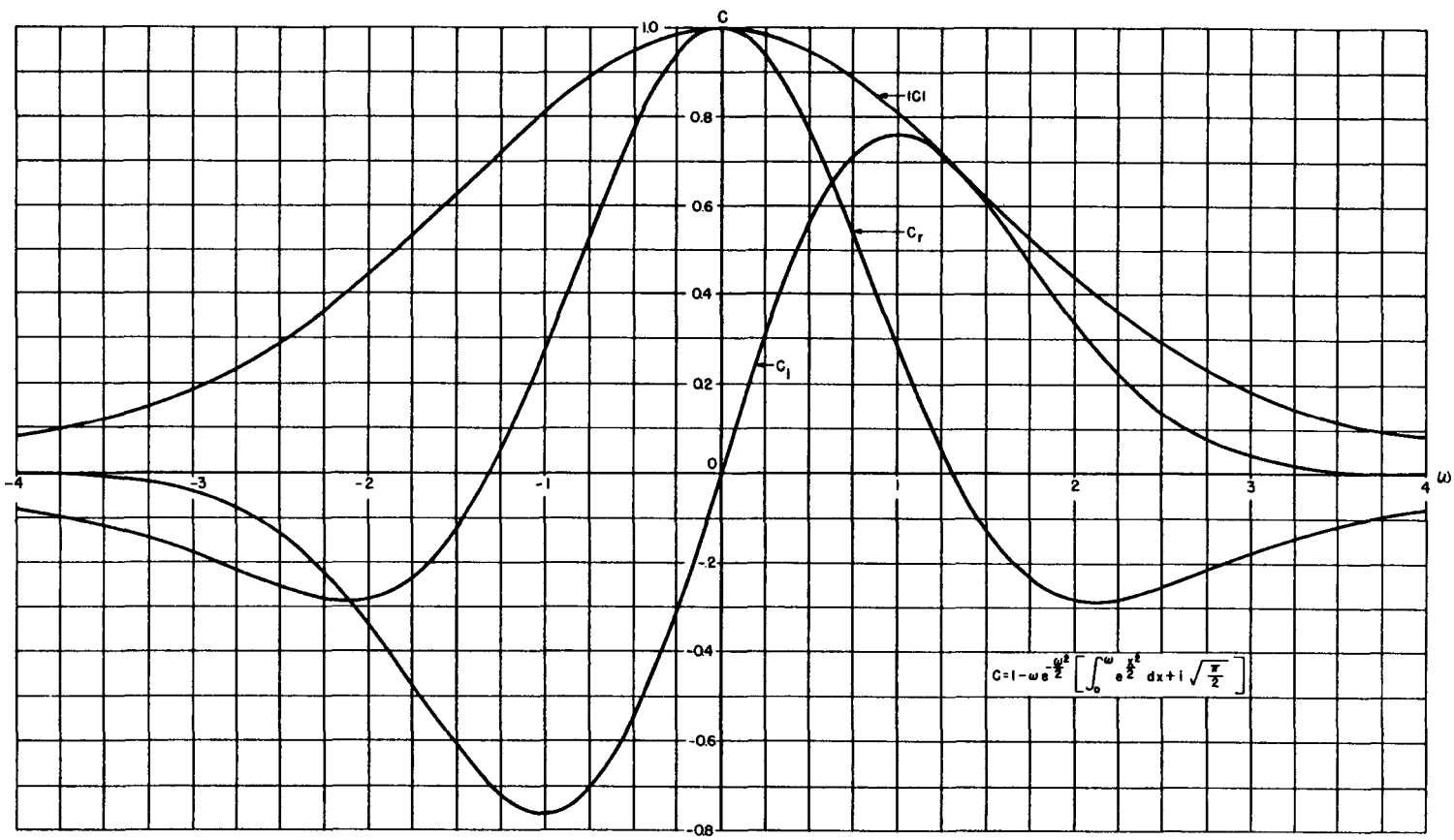
THE INCOMPLETE TORONTO FUNCTION $T_{\sqrt{q}}(2N-1, N-1, \sqrt{q})$
 FIG.30



THE INCOMPLETE TORONTO FUNCTION $T_{\sqrt{q}}(2N-l, N-l, \sqrt{q})$
 FIG.31

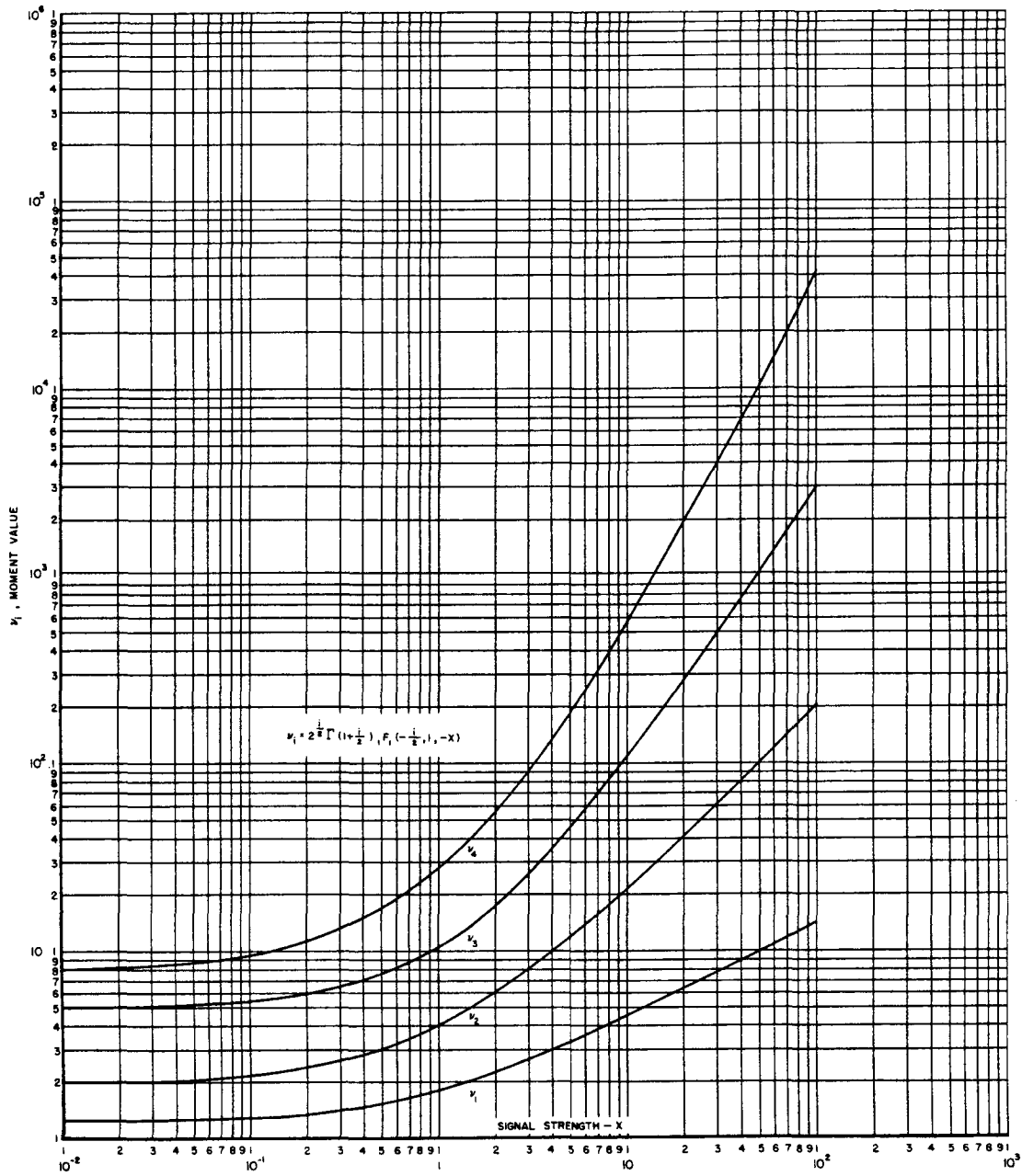


THE INCOMPLETE TORONTO FUNCTION $T_{\sqrt{v}}$ ($2N-1, N-1, \sqrt{q}$)
 FIG. 32

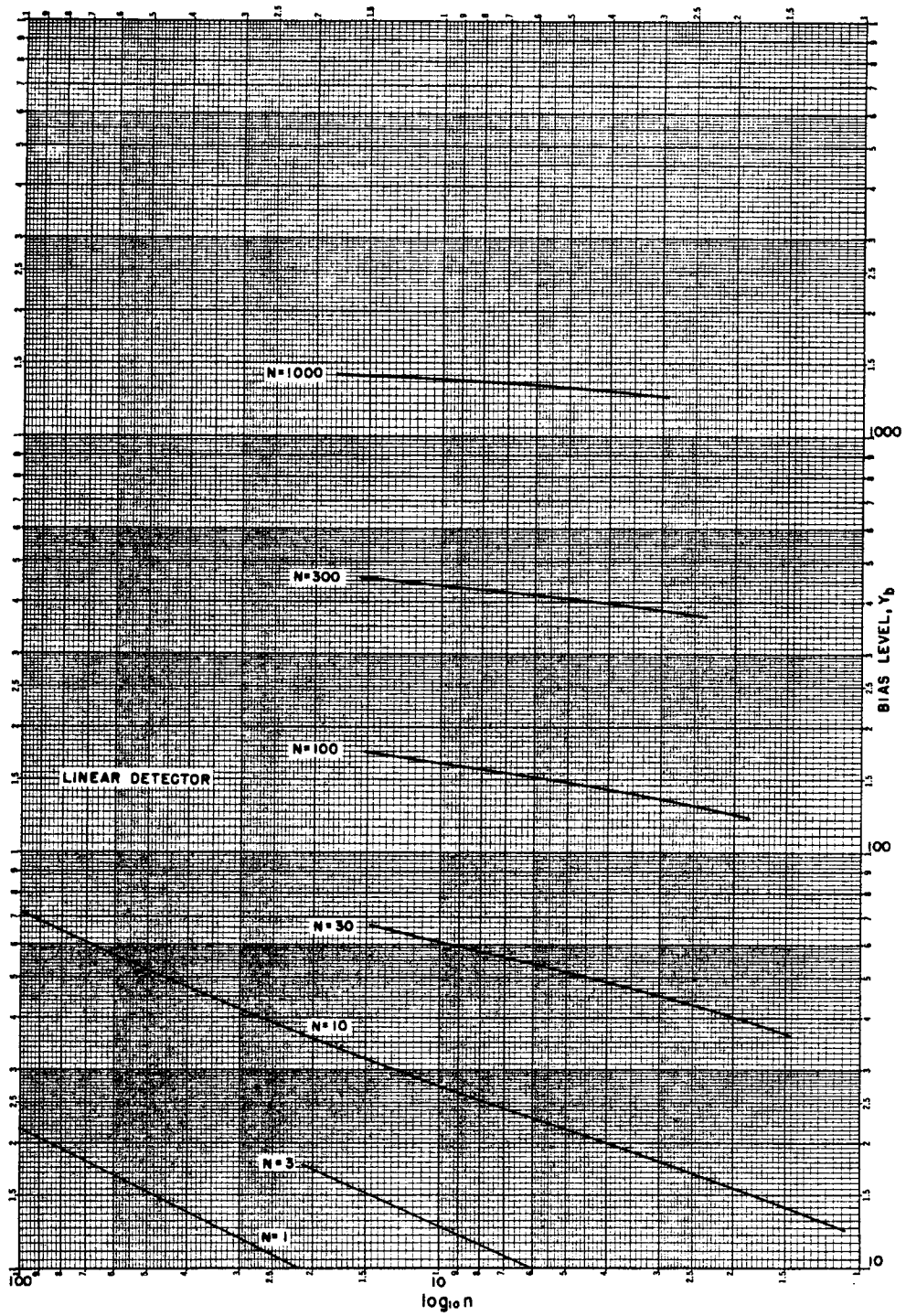


CHARACTERISTIC FUNCTION FOR ENVELOPE OF NOISE

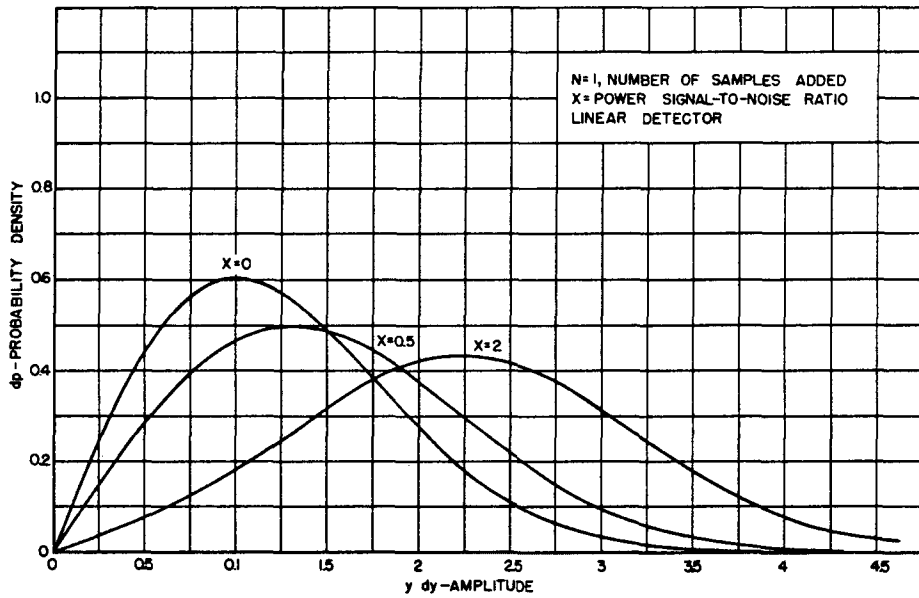
FIG. 33



MOMENTS FOR SIGNAL PLUS NOISE, LINEAR DETECTOR, N=1
 FIG 34

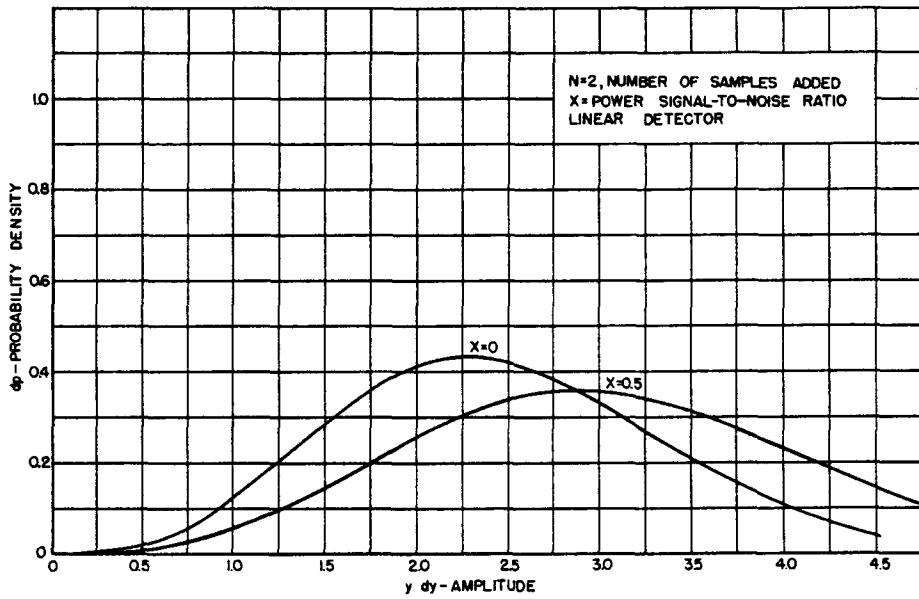


BIAS LEVEL AS A FUNCTION OF NUMBER OF PULSES INTEGRATED AND FALSE ALARM NUMBER
 FIG. 35



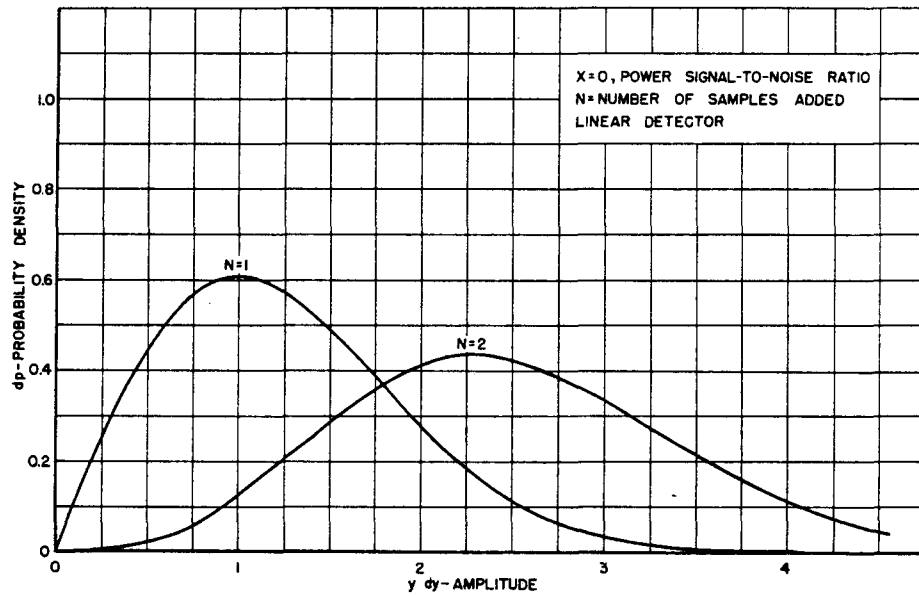
PROBABILITY DENSITY FUNCTIONS FOR ENVELOPE OF SINE WAVE PLUS NOISE

FIG.36

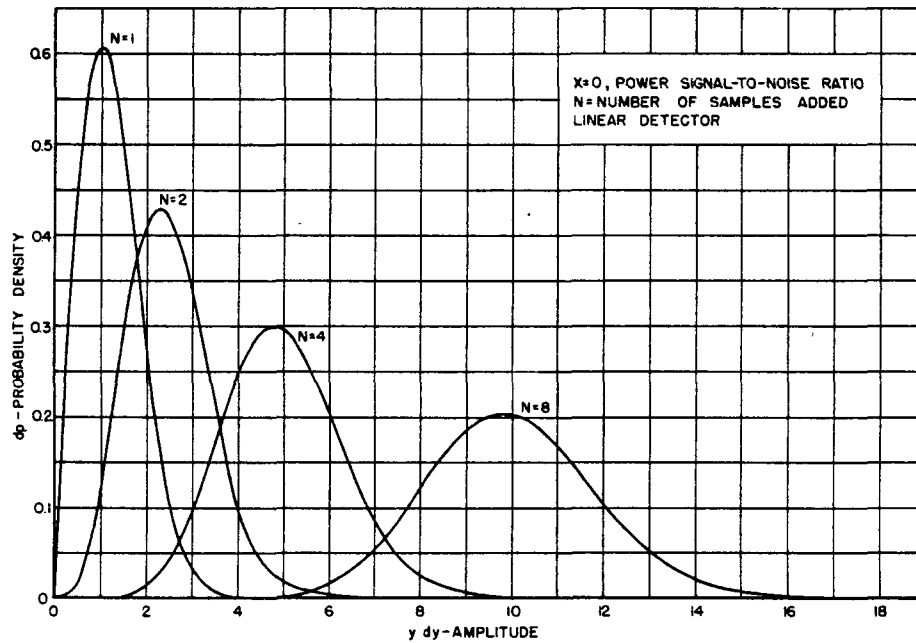


PROBABILITY DENSITY FUNCTIONS FOR ENVELOPE OF SINE WAVE PLUS NOISE

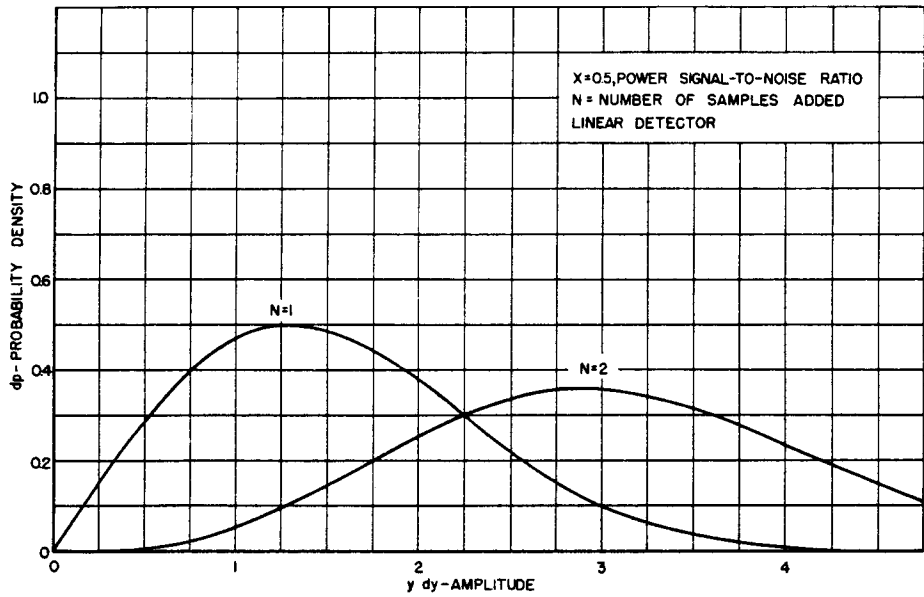
FIG.37



PROBABILITY DENSITY FUNCTIONS FOR ENVELOPE OF SINE WAVE PLUS NOISE
FIG. 38

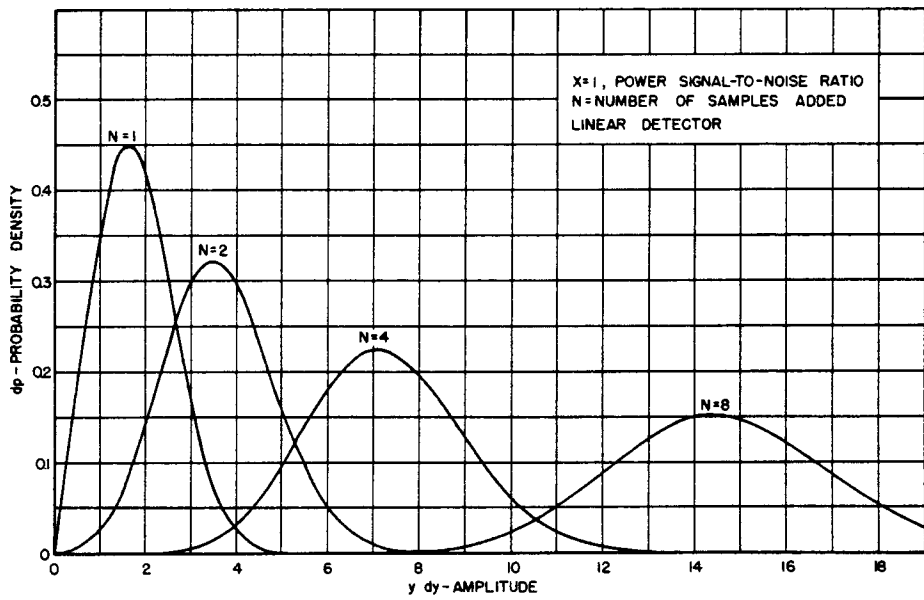


PROBABILITY DENSITY FUNCTIONS FOR ENVELOPE OF SINE WAVE PLUS NOISE
FIG. 39



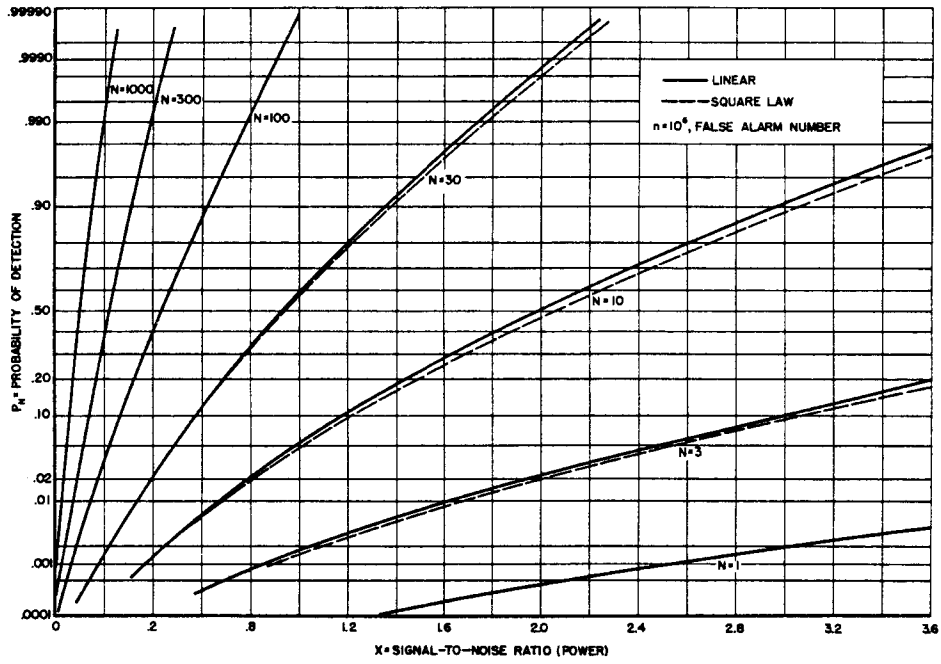
PROBABILITY DENSITY FUNCTIONS FOR ENVELOPE OF SINE WAVE PLUS NOISE

FIG. 40



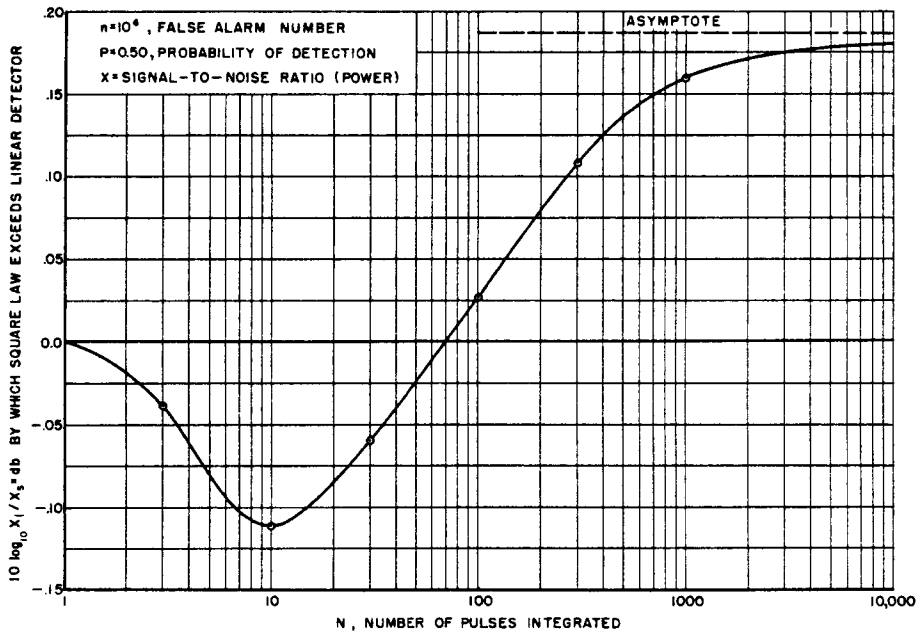
PROBABILITY DENSITY FUNCTIONS FOR ENVELOPE OF SINE WAVE PLUS NOISE

FIG. 41



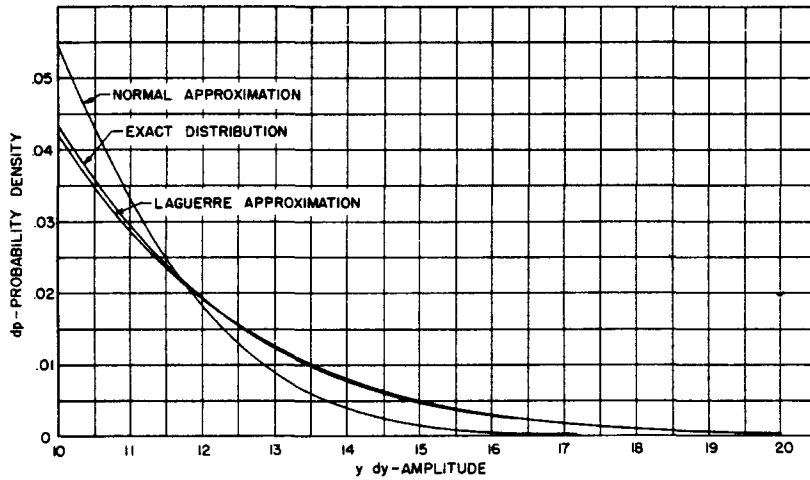
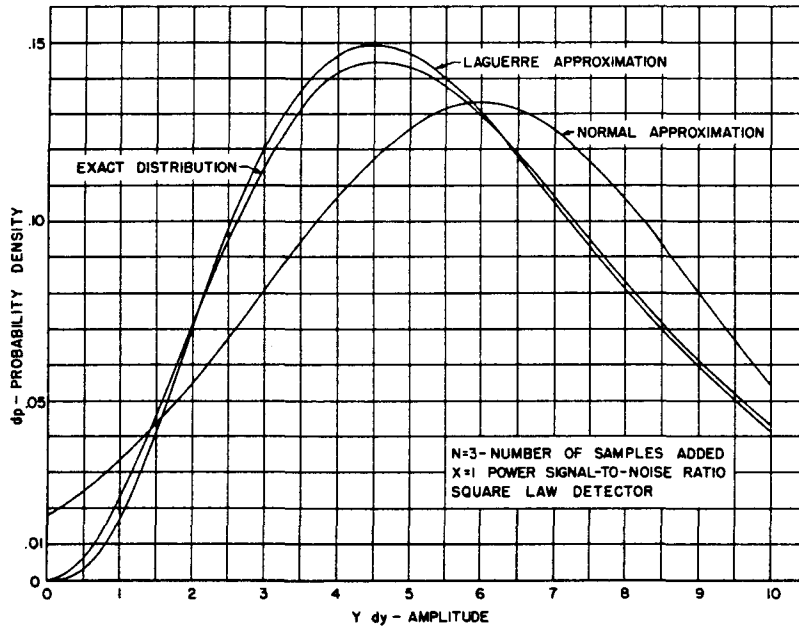
COMPARISON OF LINEAR AND SQUARE LAW DETECTORS

FIG. 42

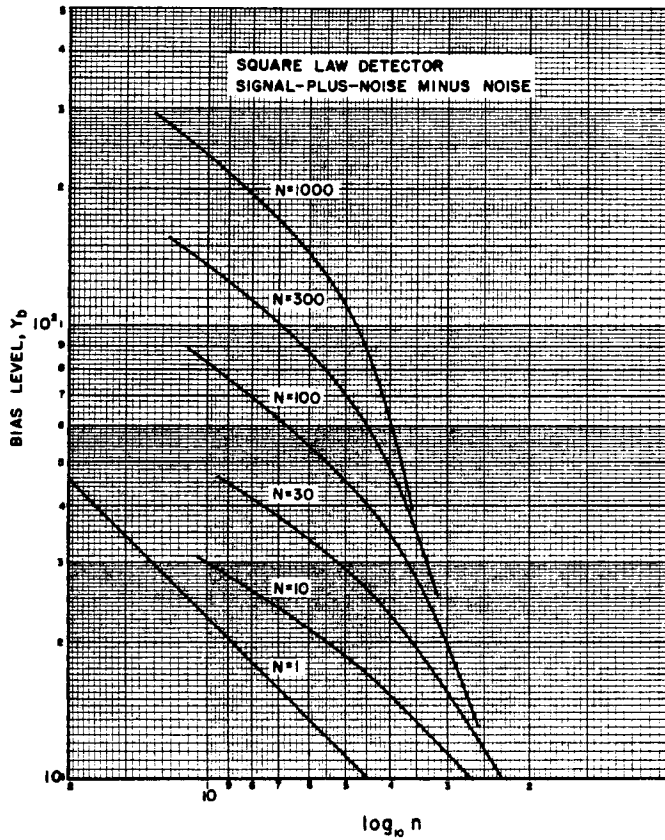


COMPARISON OF LINEAR AND SQUARE LAW DETECTORS

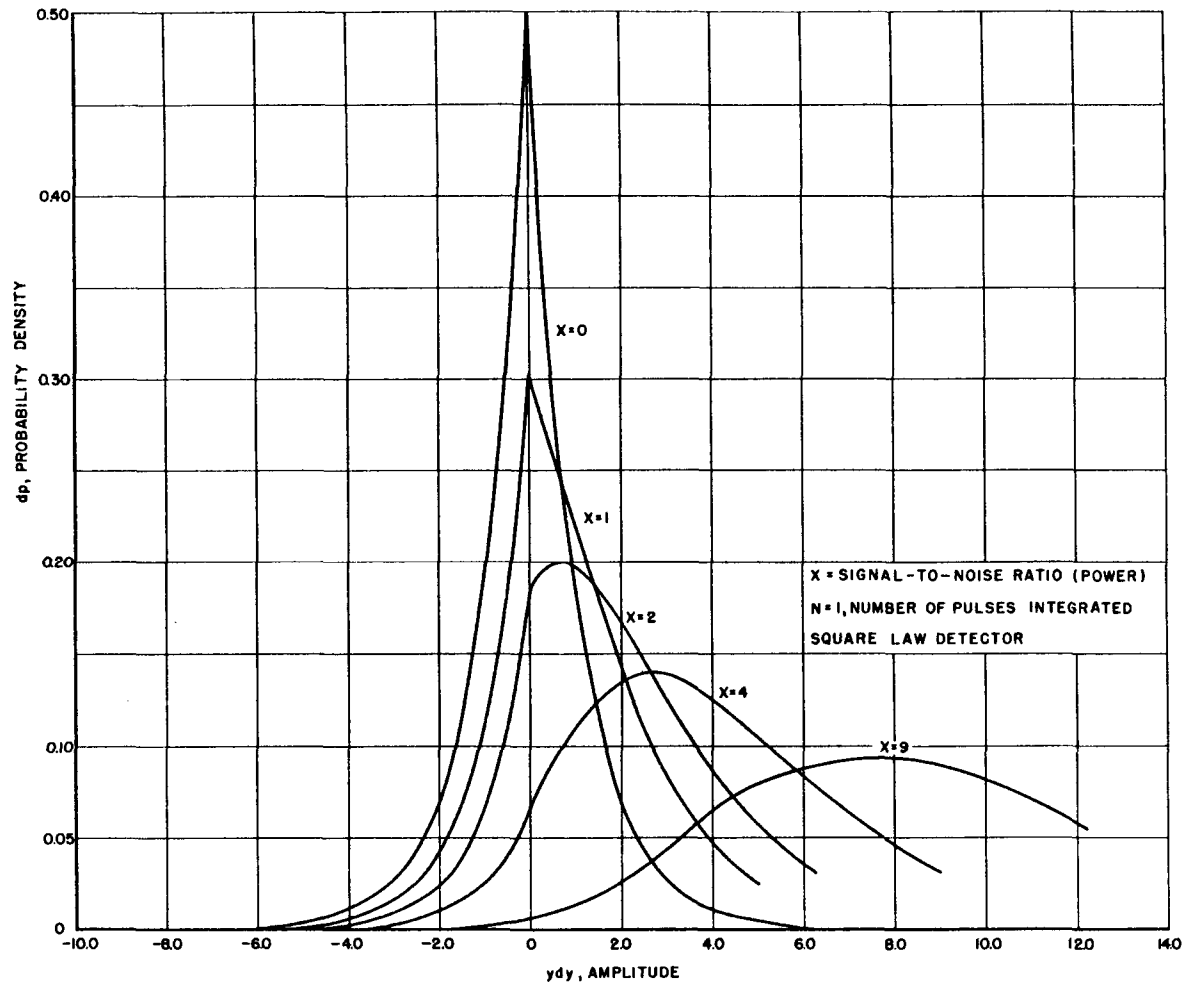
FIG. 43



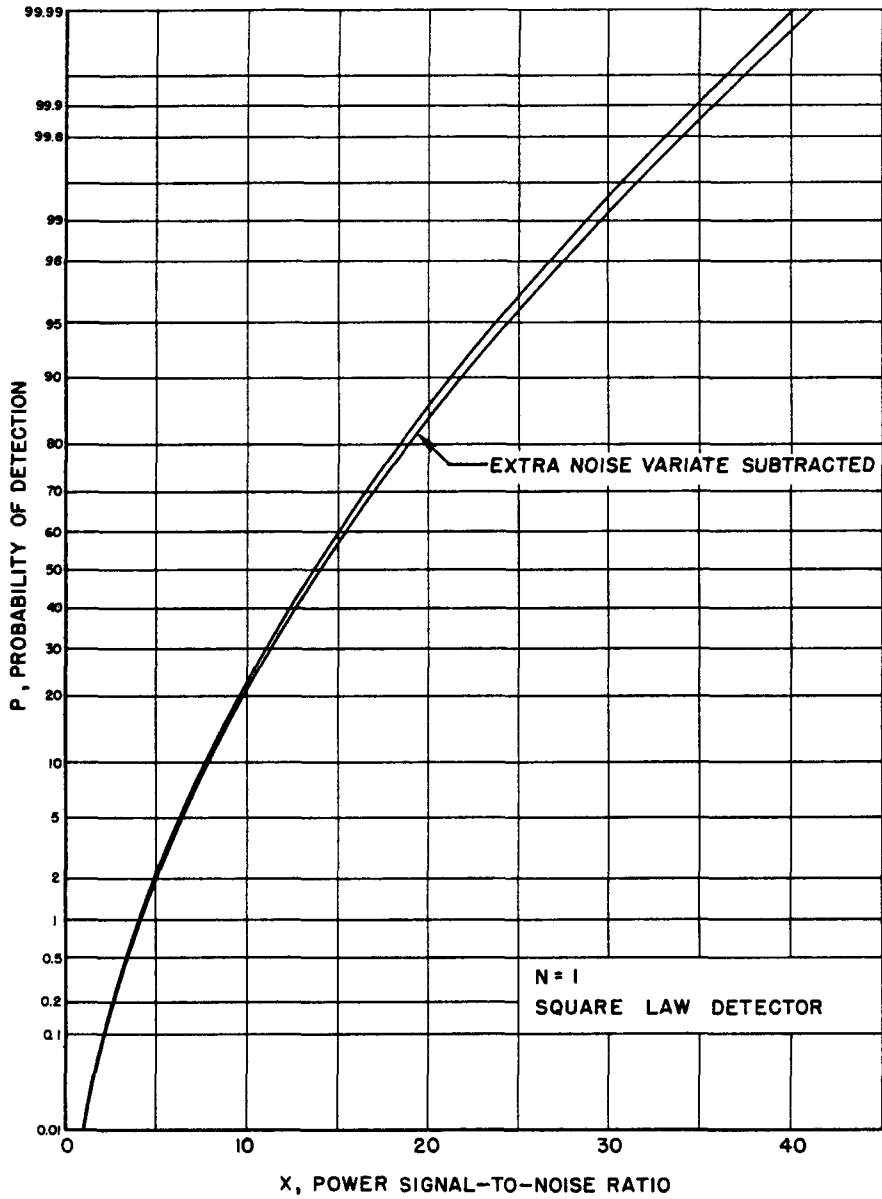
FIRST APPROXIMATIONS TO THE PROBABILITY DENSITY FOR SIGNAL PLUS NOISE
 FIG. 44



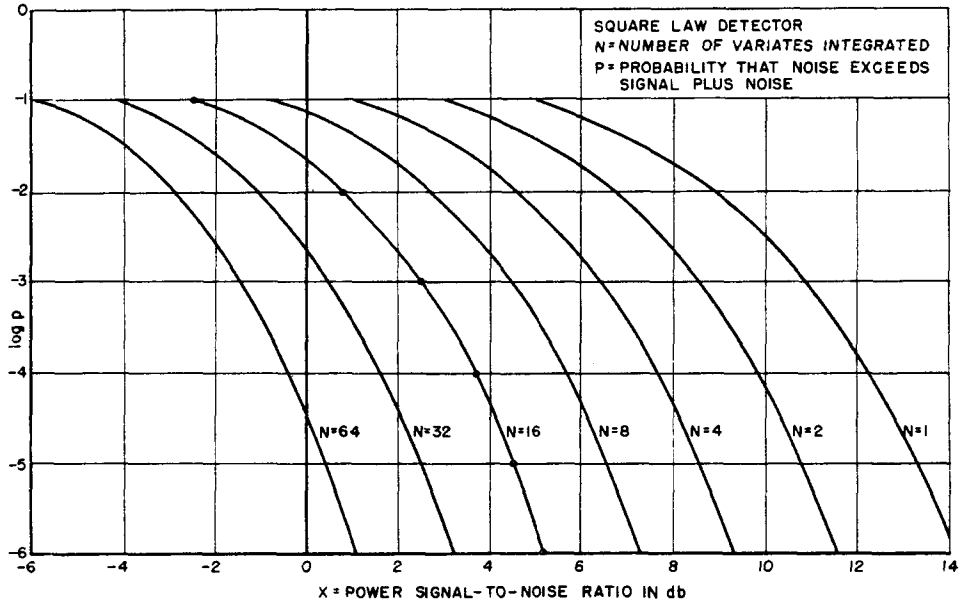
BIAS LEVEL AS A FUNCTION OF NUMBER OF
PULSES INTEGRATED AND FALSE ALARM NUMBER
FIG. 45



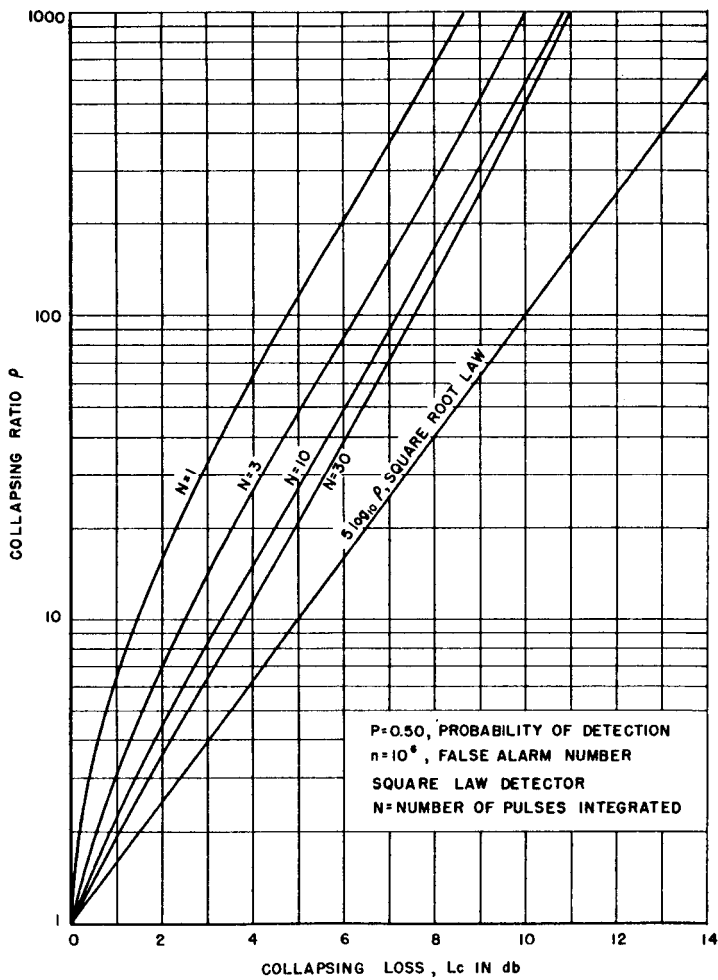
PROBABILITY DENSITY FUNCTIONS FOR A SIGNAL-PLUS-NOISE VARIATE MINUS A NOISE VARIATE
FIG.46



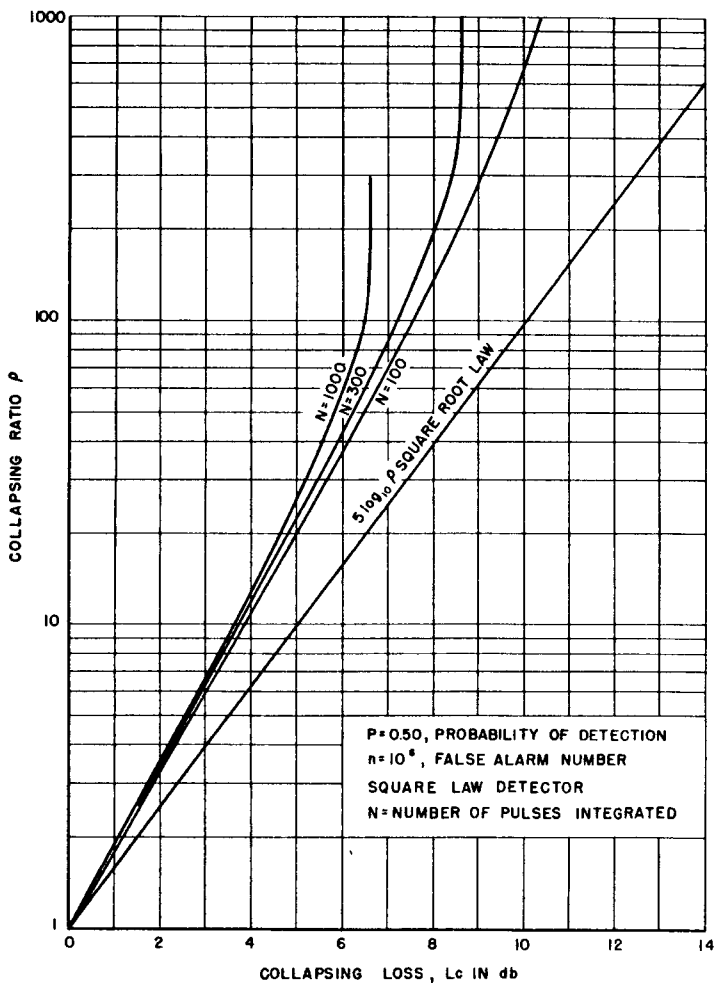
COMPARISON OF PROBABILITY OF DETECTION WHEN A NOISE VARIATE IS SUBTRACTED FROM THE SIGNAL-PLUS-NOISE VARIATE
 FIG.47



PROBABILITY THAT NOISE EXCEEDS SIGNAL-PLUS-NOISE
 FIG.48

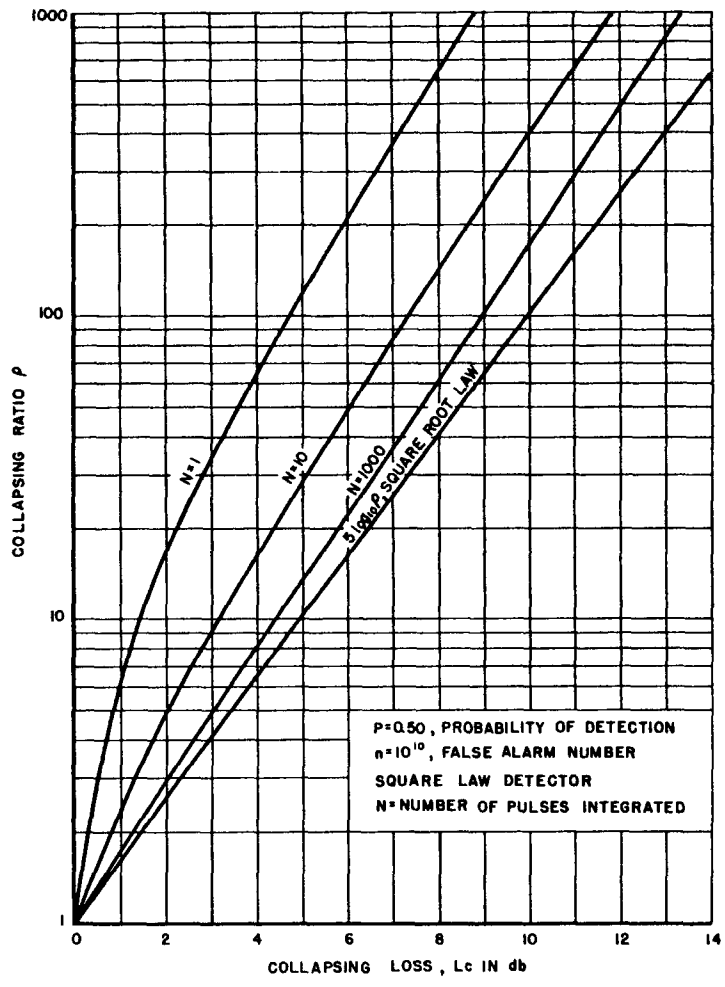


COLLAPSING LOSS FOR CONSTANT n
 FIG.49

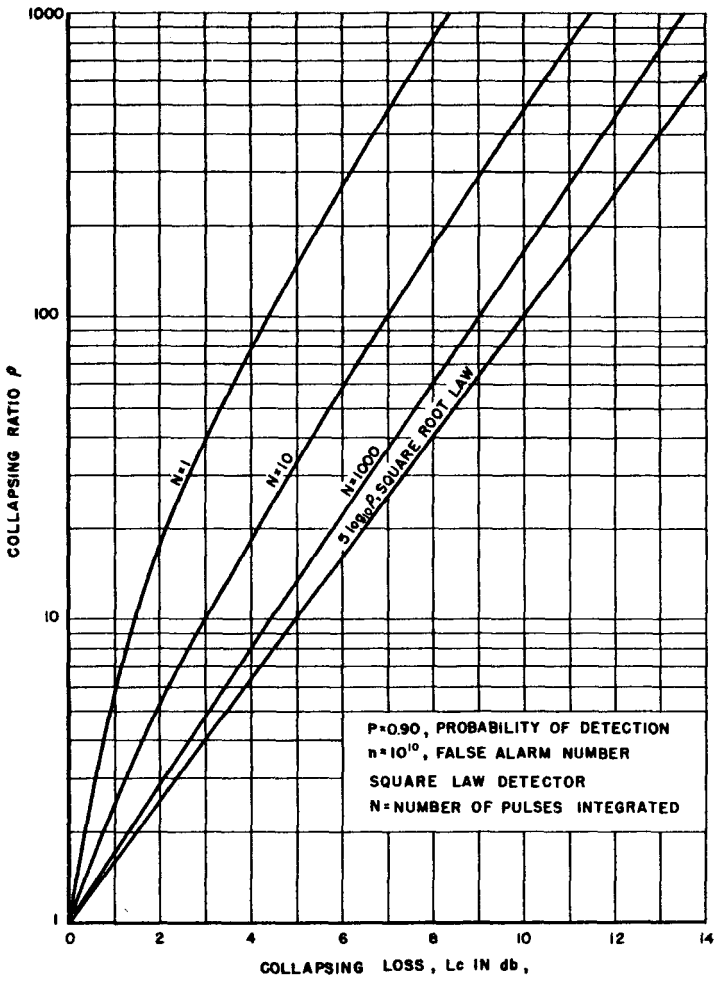


COLLAPSING LOSS FOR CONSTANT n

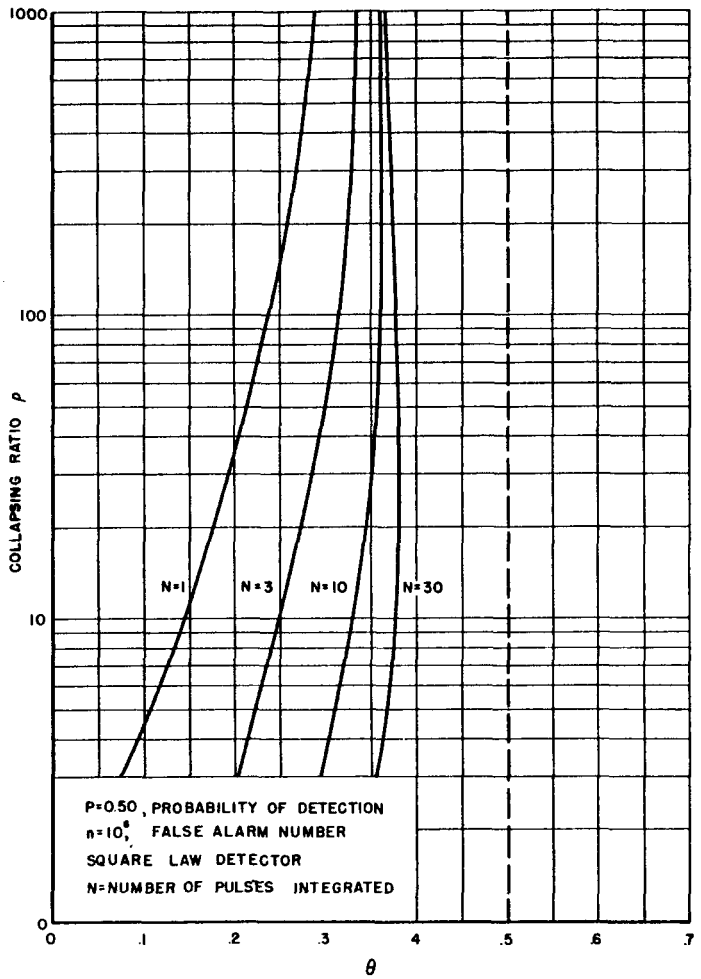
FIG. 50



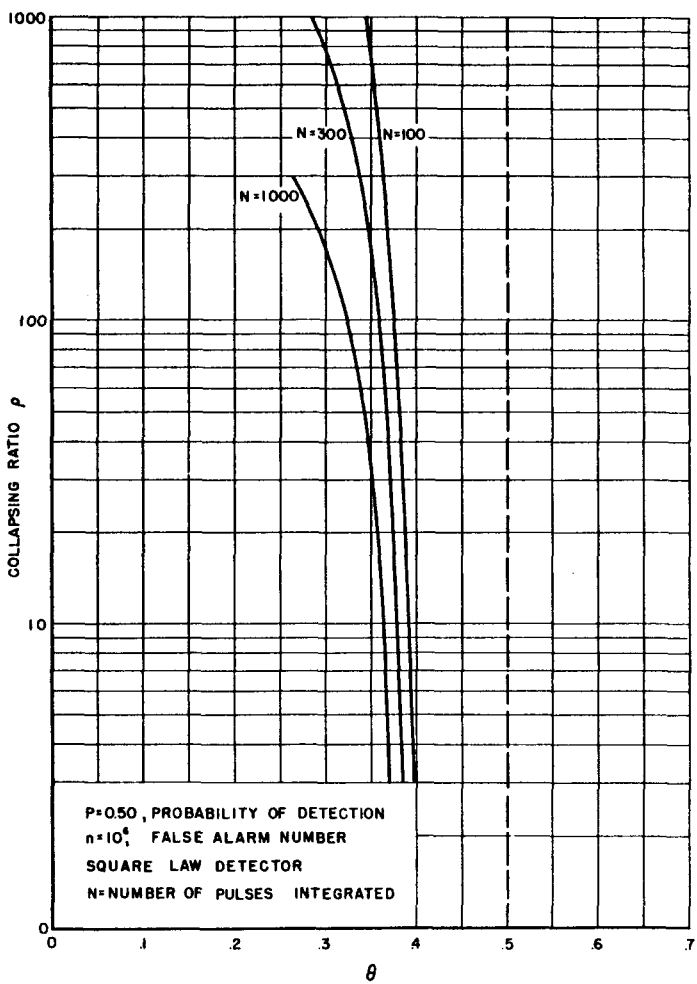
COLLAPSING LOSS FOR CONSTANT n
 FIG.51



COLLAPSING LOSS FOR CONSTANT n
 FIG. 52

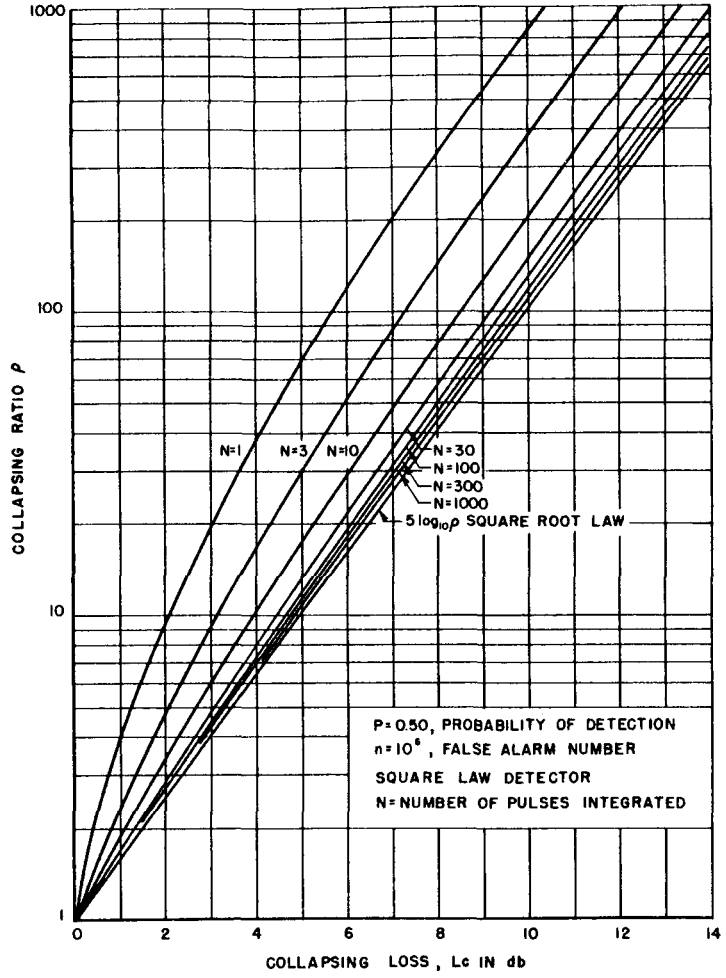


COLLAPSING LOSS FOR CONSTANT n
 FIG. 5.3



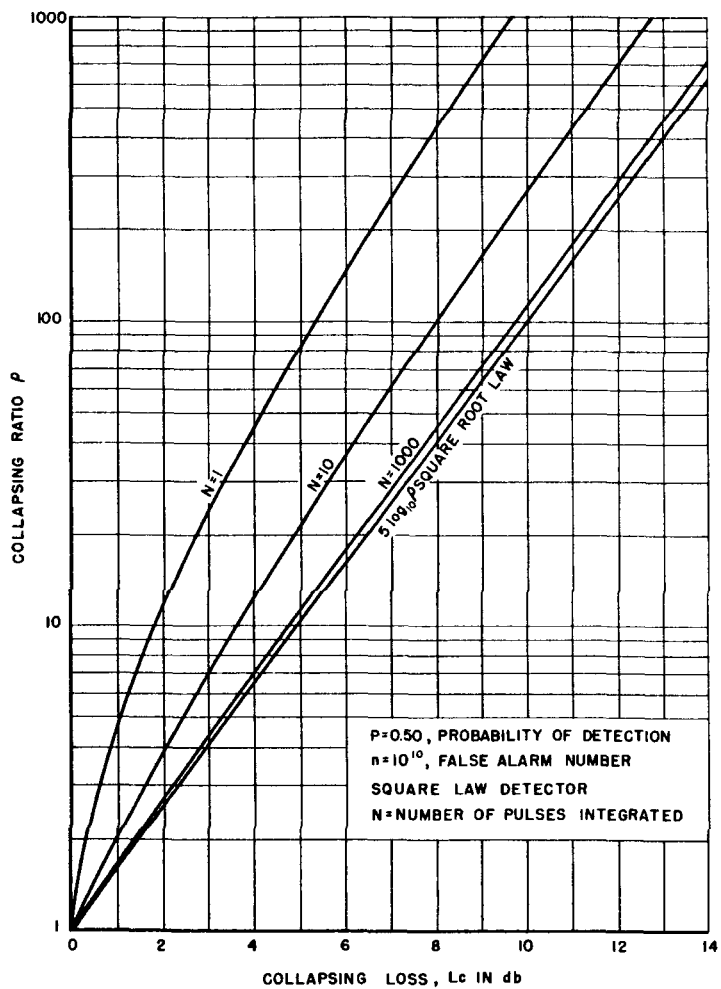
COLLAPSING LOSS FOR CONSTANT n

FIG. 54



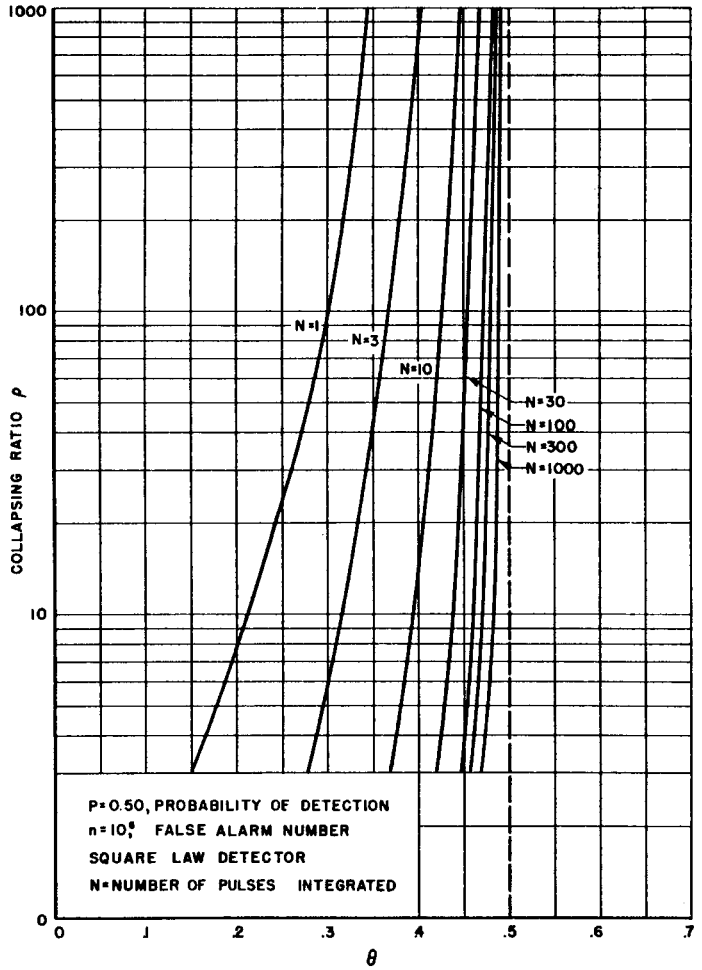
COLLAPSING LOSS FOR CONSTANT n^1

FIG. 55

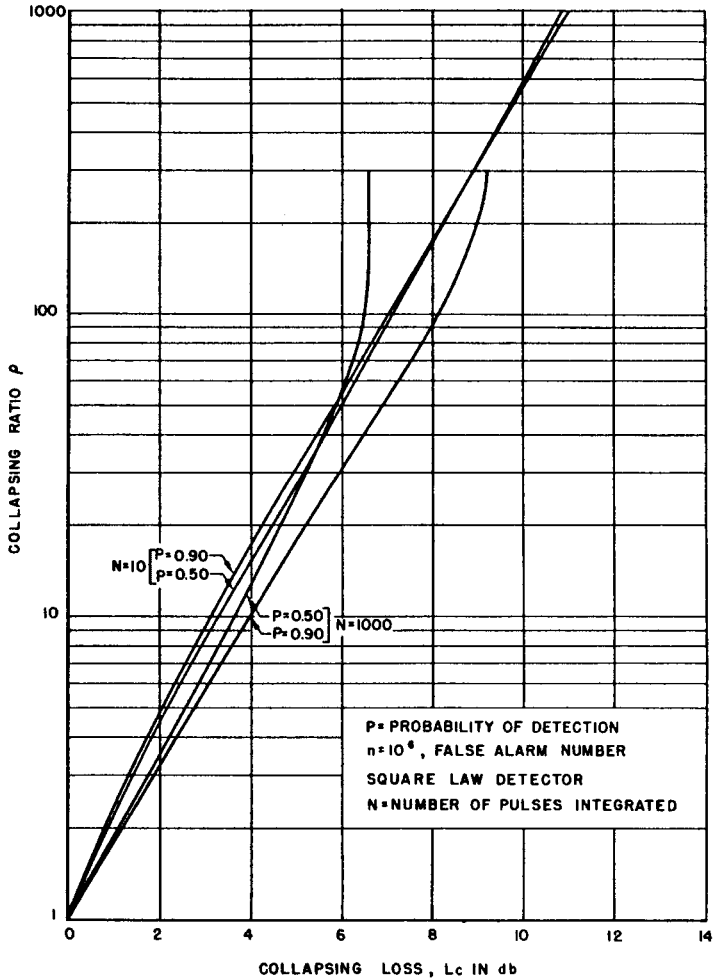


COLLAPSING LOSS FOR CONSTANT n^1

FIG. 56



COLLAPSING LOSS FOR CONSTANT n^1
 FIG.57



COLLAPSING LOSS FOR CONSTANT n
 FIG.58

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