U. S. AIR FORCE

PROJECT RAND

RESEARCH MEMORANDUM

A PURSUIT GAME WITH INCOMPLETE INFORMATION

R. P. Issacs

RM-791

11 March 1952

Assigned to ______________________

This is a working paper. It may be expanded, modified, or withdrawn at any time. The views, conclusions, and recommendations expressed herein do not necessarily reflect the official views or policies of the United States Air Force.

The RAND Corporation
1700 MAIN ST. • SANTA MONICA • CALIFORNIA

Copyright 1952
The RAND Corporation

A PURSUIT GAME WITH INCOMPLETE INFORMATION

Introduction

The game analyzed herein is the simplest non-trivial example of a pursuit game lacking full information which we could devise. The existence of this work was mentioned in our report "Games of Pursuit" (P-257). We intend to expand this paper into a junior size treatise; the present report will comprise a chapter.

The rules of this "Three-Point Signal Game" will be found in the opening sentences of the report proper. The pursuer's mode of gleaning information about the evader's position is intended to simulate the signals from an imprecise position detector such as radar. Quintessential is the pursuer's full utilization of his knowledge: his moves are governed by the information culled from the aggregate of all past signals. This aspect being our quarry, the avoidance of undue complexity forces us to rather vapid modes of moving. Both players hop from one to another of a line of points, P being allowed the greater span. Thus the essential deterrent to E's precipitate capture is P's skipping over him. Such is admittedly unrealistic, but on a one-dimensional course there is hardly an alternative.

As we stated in P-257, the play decomposes into the direct and stationary phases. The latter entails the novel signifiance and claims nearly all our attention. A new concept also emerges - of what generality it is too early to say. This is a closed set of positions such that when one is extant P can always so move as either to attain capture or maintain the position within the set. As capture is imminent at all set positions, alternatives on P's part appear strategically poor. We have assumed that they may be discarded.

The intricacy of the solution is surprising. If the reader will glance at the last pages he will see that it involves complex fractions reminiscent of some phases of number theory. Yet the graph of the part of the solution of which these are data is nearly straight. The lesson may be that simplified approximations pay off in games of this kind.

The Three-Point Signal Game

The pursuer P and the evader E each move on a discrete lineal set. We can describe a point by an integer n. If P is at n, he can make any of the four moves which carry him to $n \pm 1$, $n \pm 2$. If E is at n he can make any of three moves which carry him to n, $n \pm 1$. The players move in turn. Capture occurs when P and E occupy the same point. The payoff is the number of moves of P until capture.

After each of E's moves, P is presented with a set of three consecutive points and the information that E is on one of them with equal probabilities. More precisely: The move following E's, is a chance one with the outcome (if E occupies n) one of the three signals:

$$\sigma_{-1}(n) = (n-2,\ n-1,\ n)$$
$$\sigma_{0}(n) = (n-1,\ n,\ n+1)$$
$$\sigma_{1}(n) = (n,\ n+1,\ n+2)$$

each with probability $\frac{1}{3}$. The $\sigma_j(n)$ which arises is told to P.

To make the game fully determined we should also make the first move a chance one and its outcome E's initial position. The probabilities for E's landing on the various points are known to P.

Also P is informed when capture occurs. Beyond what has been enumerated, P has no information. On the other hand, E has full information.*

* Even the knowledge of the signal. This is, of course, unrealistic when the prototype is some unprecise position detector such as radar. But our approach is greatly simplified by making E privy to the signal.

As usual with games of incomplete information, mixed strategies will be entailed. This will mean the value of the game will actually be the expected value of the number of moves to capture.

Let us suppose both players employ optimum mixed strategies. From his knowledge of all past signals and his knowledge of E's strategy, P can garner more information of E's position than the bald amount supplied by the rules. His optimum strategy should exploit all his knowledge. Likewise E knows the extent of this knowledge and such is a factor in his decision.

We can pass at once to the stationary phase of the game. The two positions diagrammed in Figures 1 and 2 will be called the <u>first</u> and the <u>second</u>. In both cases, E has the next move.

In the first position (Fig. 1) P knows, utilizing all information at his disposal, that E occupies one of the consecutive points on the same side of him with probabilities S and 1-S.* ($0 \leq S \leq 1$). In the second, P knows that E as at one of the one of two points adjacent to him, with probabilities T and 1-T. (Fig. 2), here $0 \leq T \leq 1$, but we can, by symmetry, suppose $0 \leq T \leq \frac{1}{2}$.

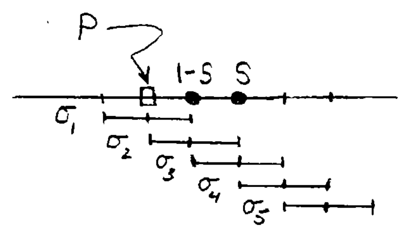

Fig. 1

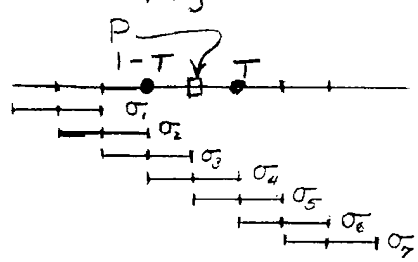

Fig. 2

These two positions constitute a closed set in the following sense: Let E follow the dictates of some definite (and known to P) mixed strategy. Let a cycle of move occur, i.e., E moves, there is a chance move yielding a signal, P moves. Then we can directly verify that P can always so select his move that either: a) capture occurs or, b) the first or second position

---

* Really there is a set of first positions--one for each S.

occurs (or recurs) although in general with a new value of S or T.

Now the moves of P which do not lead to a) or b) appear to be poor ones. We therefore make at once the plausible assumption that they are excluded from the optimal strategies of P and consider them no further.*

Should P and E be initially far apart, it is clear that elementary tactics on the part of P will ultimately bring about a first or second position (or capture).** Thus we are justified in speaking of the present phase of the game as the stationary one.

What we are doing may be regarded as starting anew with a fresh class of games. For example, in the first position case, the first move is a chance one of probability S (S known to both players). E moves next, etc.

We wish to calculate $\mathcal{E}(S)$ and $\mathcal{F}(T)$, the expected number of moves to capture for the first and second positions. A simple example from classic probability theory will clarify our technique. Suppose a coin is tossed until it lands heads. What is the expected number of tossings? Calling it e, the first toss has two outcomes, each with probability $\frac{1}{2}$. If heads, there is one toss; if tails, we revert to the initial situation and the expected number is 1 + e.

Thus

$$e = \frac{1}{2} \cdot 1 + \frac{1}{2}(1 + e),$$

an equation which may be solved to obtain e.

Let us turn to the first position. The data furnished P is a value of S and one of the five σ_j shown in Fig. 1. He has the choice of moving one or two points to the right. We thus can tabulate his strategy:

* The analysis of most reasonably complex games is encumbered by a profusion of manifestly absurd strategies. If they be discarded early, subsequent development becomes greatly streamlined.

** This claim will be definitely substantiated later.

	1	2
1	1	0
2	$P_2(S)$	$1-P_2(S)$
3	$P_3(S)$	$1-P_3(S)$
4	0	1
5	0	1

The upper indices are the number of moves; those on the left, the j of the σ_j received. The entries are probabilities. We have suppressed certain manifestly foolish choices. For example, if P receives σ_1, he knows E's position and captures at once by moving one space. Likewise two spaces is clearly the move in the event of σ_4 or σ_5. If σ_2 is received he jumps one space with probability $P_2(S)$ $(0 \leq P_2 \leq 1)$, etc.

We could form a similar table of E's strategies. But, instead, we will combine E's move with the chance move which precedes it (i.e., the move which locates E with probabilities S and 1-S) and deal only with the resultant probabilities. In Fig. 1, suppose P is at n; then after E's move, E will be at either n + 1, n + 2, or n + 3. Let the respective probabilities, as a result of the two moves combined, of E's being at these points be A(S), B(S), C(S). What control does E enjoy over these numbers? Clearly he cannot get to n + 3 unless he started from n + 2; thus $C(S) \leq S$. We shall show this is the only restriction.

Lemma 1. Let S, A, B, C be numbers in $[0,1]$ with A + B + C = 1 and $C \leq S > 0$. Then if the chance move suffers probability value S, E can select his strategy so that the net probabilities of his landing on the three points to the right of E are A, B, C.

Proof. If the first move puts E at n + j (j = 1,2) let the probabilities of his moving -1, 0, 1 points to the right be $Q_{j,-1}$, $Q_{j,o}$, $Q_{j,1}$. Clearly

$Q_{1,-1} = 0$. We have to show that the Q_{jk} can be chosen to satisfy

$$\begin{aligned}
(1 - S)\, Q_{10} + S\, Q_{2,-1} &= A \\
(1 - S)\, Q_{11} + S\, Q_{2,0} &= B \\
S\, Q_{21} &= C \\
Q_{10} + Q_{11} &= 1 \\
Q_{2,-1} + Q_{20} + Q_{21} &= 1 \\
0 \leq Q_{jk} &\leq 1
\end{aligned}$$

If $S = 1$, a solution is evident. If $S < 1$ let λ be any number satisfying

$$\left.\begin{aligned}
&0 \leq \lambda \leq 1 \\
&\frac{1 - B - S}{1 - S} \leq \lambda \leq \frac{A}{1 - S}.
\end{aligned}\right\} \quad (1)$$

This is possible as $\frac{1 - B - S}{1 - S} \leq 1$, $\frac{A}{1-S} \geq 0$, and $\frac{1-B-S}{1-S} \leq \frac{1-B-C}{1-S} = \frac{A}{1-S}$.

Putting

$$\left.\begin{aligned}
Q_{10} &= \lambda \\
Q_{11} &= 1-\lambda \\
Q_{2,-1} &= \frac{A-(1-S)\lambda}{S} \\
Q_{20} &= \frac{B - (1-S)(1-\lambda)}{S} \\
Q_{21} &= \frac{C}{S}
\end{aligned}\right\} \quad (2)$$

we can verify that all of our conditions are satisfied.

Remark: We shall solve our modified problem and $A(S)$, $B(S)$, $C(S)$, shall be a part of our answer. If the solutions in the original terms is wanted, the partial strategies of E (i.e., the Q_{jk}) may be ascertained from (2) with λ taken as any function of S satisfying (1).

We proceed as follows with the first position. We list all possible outcomes after a cycle of moves, multiply them by their probabilities and by their subsequent state (as in the coin tossing problem), and add. For example: suppose E, after moving, is at n + 2, P receives the signal σ_3, and P moves one space. The probability of all this happening is

$$B(S) \cdot \frac{1}{3} \cdot P_3(S).$$

A first position results. But the value of S is now $\frac{C(S)}{B(S) + C(S)}$. Thus this possibility will contribute the term

$$\frac{1}{3} B(S) P_3(S) \left[1 + \mathcal{E}\left(\frac{C(S)}{B(S) + C(S)}\right)\right].$$

We write the full expression, omitting the arguments S on the right side for brevity:

$$\mathcal{E}(S) = \frac{1}{3} \; A\left[1 + P_2 \cdot 1 + (1-P_2)(1+\mathcal{E}(0)) + P_3 \cdot 1 + (1-P_3)(1+\mathcal{F}(\tfrac{C}{A+C}))\right]$$

$$+ B\left[P_2(1+\mathcal{E}(0)) + (1-P_2)\cdot 1 + P_3 \; (1+\mathcal{E}(\tfrac{C}{B+C})) + (1-P_3)\cdot 1 + 1\right]$$

$$+ C\left[P_3(1+\mathcal{E}(\tfrac{C}{B+C})) + (1-P_3)(1 + \mathcal{F}(\tfrac{C}{A+C})) + 2 \; (1 + \mathcal{E}(0))\right]$$

$$= 1 + (A+2C)\frac{1}{3}\,\mathcal{E}(0) + \frac{(A+C)}{3}\mathcal{F}(\tfrac{C}{A+C}) + P_2 \,\frac{\mathcal{E}(0)}{3}\left[B - A\right] + P_3$$

$$\left[-\frac{1}{3}\,(A + C)\mathcal{F}(\tfrac{C}{A+C}) + \frac{1}{3}\,(B + C)\mathcal{E}(\tfrac{C}{B+C})\right]. \qquad (3)$$

Let us call the right side of (3), Ψ. If, in Ψ the A, B, C, P_2, P_3 are considered as prescribed functions of S, (3) concerns itself with expressing the expected number of moves to capture in terms of them. However, we are after optimum strategies.

Let us call the right side of (3) $\Psi(A,B,C,P_2,P_3,[\mathcal{E},\mathcal{F}])$.* In writing (3) we considered $A,\ldots,P_3$ as given functions of S. Then (3) is a functional equation satisfied by the $\mathcal{E}$ and $\mathcal{F}$ which arise when these definite strategies are employed. (Of course, there is a similar equation for the second position. We will exhibit it shortly.)

But our goal is optimal play. Let $\mathcal{E}^*$ and $\mathcal{F}^*$ denote the outcomes when both players employ optimal strategies. Think, for the moment, of S as fixed. Then (3) is descriptive of a transition from one fixed position to a number of possible others. Suppose that for all these others the optimal values have already been ascertained. We think of $A,\ldots,P_3$ as a set of numbers and, if these have been chosen, we have for the original position,

$$\mathcal{E}(S) = \Psi(A,\ldots,P_3,[\mathcal{E}^*,\mathcal{F}^*]).$$

We now apply the tenet of transition. We obtain

$$\mathcal{E}^*(S) = \max_{\substack{A,B,C\\ C\leq S}} \ \min_{P_2,P_3} \ \Psi(A,\ldots,P_3,[\mathcal{E}^*,\mathcal{F}^*]). \qquad (4)$$

This and a later one like it for the second position, are the basic equations with which we work. They are functional equations for $\mathcal{E}^*$ and $\mathcal{F}^*$ and we will see later that they have a unique solution. As we will henceforth be concerned only with optimal play, we will drop the asterisks from $\mathcal{E}^*$ and $\mathcal{F}^*$. $A(S),\ldots,P_3(S)$ will denote the minimizing or maximizing values of $A,\ldots,P_3$ in (4) and describe the optimal strategies.

We can evaluate $\mathcal{E}(0)$ at once. This is because the first position with S = 0 is closed in itself. We observe the first position with S = 0 is the same as the second with T = 0 or 1.

<u>Theorem 1</u>. $\mathcal{E}(0) = 3/2$, $A(0) = B(0) = \frac{1}{2}$, $P_2(0) + P_3(0) = 1$.

* The last two arguments are an abbreviation. They symbolize all the $\mathcal{E}$ and $\mathcal{F}$ expressions of the right member of (3). In our later work, we shall omit writing them.

Remark: It is evident that C(C) must be 0 here. Furthermore the signals σ_2 and σ_3 convey exactly the same information; thus a strategy requires only the value of the arithmetic mean of P_2 and P_3 here given by

$$\frac{1}{2}\,(P_2(0) + P_3(0)) = \frac{1}{2}.$$

Proof. As C = 0, we get from (3) and (4)

$$\mathcal{E}(0) = \max_{\substack{A,B\\A+B=1}} \min_{P_2\ P_3} \left[1 + \frac{2}{3}A\,\mathcal{E}(0) + P_2\,\frac{\mathcal{E}(0)}{3}(B-A) + P_3\,\frac{\mathcal{E}(0)}{3}(B-A)\right]$$

$$= 1 + \mathcal{E}(0)\,\mathcal{J}$$

where

$$\mathcal{J} = \max_{\substack{A,B\\A+B=1}} \min_{P_2,P_3} \left[\frac{2}{3}A + \frac{(P_2+P_3)}{3}(B-A)\right] = \max\min\ \Theta(A,B,P_2,P_3)$$

$$\leq \max_{A,B}\ \Theta\left(A,B,\frac{1}{2},\frac{1}{2}\right) = \max\left[\frac{2}{3}A + \frac{(B-A)}{3}\right]$$

$$= \max_{\substack{A,B\\A+B=1}} \frac{A+B}{3} = \frac{1}{3}.$$

$$\mathcal{J} \geq \min_{P_2P_3}\ \Theta\left(\frac{1}{2},\frac{1}{2},P_2,P_3\right) = \min_{P_2,P_3}\left[\frac{1}{3}\right] = \frac{1}{3}.$$

Therefore $\mathcal{E}(0) = 1 + \frac{1}{3}\,\mathcal{E}(0)$

$$\mathcal{E}(0) = \frac{3}{2}. \qquad (5)$$

Now for the second position! The following are the strategy tables:

P:

	-2	-1	1	2
σ_1	1	0	0	0
σ_2 or σ_3	$1-p_2(T)$	$p_2(T)$	0	0
σ_4	0	$1-p_4(T)$	$p_4(T)$	0
σ_5 or σ_6	0	0	$p_5(T)$	$1-p_5(T)$
σ_7	0	0	0	1

(We note that σ_2 and σ_3, as well as σ_5 and σ_6, convey precisely the same information to P and are thus equivalent)

E:

	-1	0	1
n+1	0	$1-q_1(T)$	$q_1(T)$
n-1	$q_{-1}(T)$	$1-q_{-1}(T)$	0

(The left index is the position of E if P is at n)

Now proceeding as before: (we use $\frac{3}{2}$ in place of $\mathcal{E}(0)$)

$$\mathcal{F}(T) = \frac{1}{3}\Big\{(1-T)q_{-1}\Big[1+2((1-p_2)+p_2(1+\tfrac{3}{2}))\Big] + (1-T)(1-q_{-1})\Big[2((1-p_2)(1+\tfrac{3}{2})+p_2)$$
$$+(1-p_4)+p_4\,(1+\mathcal{E}(1))\Big] + T(1-q_1)\Big[(1-p_4)(1+\mathcal{E}(1))+p_4+2\,(p_5+(1-p_5)$$
$$(1+\tfrac{3}{2}))\Big] + Tq_1\Big[2(p_5(1+\tfrac{3}{2})+(1-p_5))+1\Big]\Big\}$$

$$= 1 + (1-T)\Big[1-q_{-1}-p_2+2p_2Q_{-1}\Big] + T\Big[1-q_1-p_5+2p_5\,Q_1\Big]$$
$$+\frac{1}{3}\,\mathcal{E}(1)\Big[T+p_4(1-2T)-q_1\,T(1-p_4)-q_{-1}p_4(1-T)\Big]$$

$$= \phi\,(T,\, q_{-1}, q_1, p_2, p_4, p_5) \qquad (6)$$

Or, for the optimal case:

$$\mathcal{F}(T) = \max_{q_{-1},q_1}\ \min_{p_2,p_4,p_5}\ \phi. \qquad (7)$$

<u>Lemma 2</u> If $0 \leq {p \atop q} \leq 1$ and $L = 1 - p - q + 2pq$

then $0 \leq L \leq 1.$

<u>Proof</u>: $L = 2(p - \frac{1}{2})(q - \frac{1}{2}) + \frac{1}{2} = 2\,PQ + \frac{1}{2}$

with ${|P| \atop |Q|} \leq \frac{1}{2}.$

$$0 = -2\,\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2} \leq L \leq 2\cdot\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2} = 1.$$

<u>Lemma 3</u> $\mathcal{F}(T) \leq 2 + \frac{\mathcal{E}(1)}{3}$

<u>Proof</u>: We may suppose $T \leq \frac{1}{2}$. In $\emptyset$, the coefficient of $\mathcal{E}(1)$ is

$$\leq \frac{1}{3}\left[T + p_4(1-2T)\right] \leq \frac{1}{3}\left[T + (1-2T)\right] = \frac{1}{3}(1-T) \leq \frac{1}{3}.$$

By Lemma 2, applied to the coefficients of T and 1 - T, all the other terms of $\emptyset$

$$\leq 1 + (1-T) + T = 2.$$

<u>Lemma 4</u> $\mathcal{E}(1) \leq 3$

<u>Proof</u>: From (3) and (4)

$$\mathcal{E}(1) \leq \max_{A,B,C} \Psi(A,B,C,0,0)$$

$$= \max_{A,B,C}\left[1 + \frac{1}{2}(A + 2C) + \frac{1}{3}(A + C)\,\mathcal{F}\left(\frac{C}{A+C}\right)\right]$$

$$= \max\left[1 + (A + C)\left(\frac{1}{3}\mathcal{F}\left(\frac{C}{A+C}\right) + \frac{1}{2}\right) + \frac{C}{2}\right]$$

$$= \max_{A,B,C} e_1(A,C)$$

By Lemma 3, for every A and C

$$e_1(A,C) \leq 1 + (A + C)\left(\frac{2}{3} + \frac{\mathcal{E}(1)}{9} + \frac{1}{2}\right) + \frac{C}{2}$$

$$\leq 1 + \left(\frac{2}{3} + \frac{1}{2} + \frac{\mathcal{E}(1)}{9}\right) + \frac{1}{2} = \frac{8}{3} + \frac{\mathcal{E}(1)}{9}$$

Thus $\mathcal{E}(1) \leq \frac{8}{3} + \frac{\mathcal{E}(1)}{9}$

$$\frac{8}{9}\ \mathcal{E}(1) \leq \frac{8}{3}$$

$$\mathcal{E}(1) \leq 3.$$

<u>Theorem 2</u>. If $T \leq \frac{1}{2}$

$$\mathcal{F}(T) = \frac{3}{2} + \frac{\mathcal{E}(1)}{6}\,T \quad \left[= \frac{3}{2} + \frac{5}{12}\,T \right]$$

and the optimum strategies are given by:

If $T \leq \frac{1}{2}$

$$q_1 = q_{-1} = \frac{1}{2}\ ;$$

if $T < \frac{1}{2}$

$$p_2 = \frac{1}{2},\ p_4 = 0,\ p_5 = \frac{1 + \frac{\mathcal{E}(1)}{3}}{2} \quad \left[= \frac{11}{12} \right]\ ;$$

if $T = \frac{1}{2}$

$$p_2 = p_5 = \frac{1 + \frac{\mathcal{E}(1)}{6}}{2} \quad \left[= \frac{17}{24} \right],\ p_4 = \frac{1}{2}\ .$$

<u>Remark 1</u>. The parts of the above statement in square brackets are not intended as part of the theorem. Our next result will show that $\mathcal{E}(1) = \frac{5}{2}$; then the brackets follow at once.

<u>Remark 2</u>. Lemma 4 assures us that (in both cases) $p_5 \leq 1$.

<u>Remark 3</u>. If $T > \frac{1}{2}$ we utilize evident symmetries to find $\mathcal{F}(T)$ and the strategies.

Proof:

1) $T < \frac{1}{2}$

$$\mathcal{F}(T) \geq \min_{p_2, p_4, p_5} \phi(T, \tfrac{1}{2}, \tfrac{1}{2}, p_2, p_4, p_5)$$

$$= \min \left\{ 1 + (1-T)\frac{1}{2} + T\frac{1}{2} + \frac{1}{3}\mathcal{E}(1)\left[\frac{T}{2} + p_4\left(\frac{1}{2} - T\right)\right]\right\}$$

The minimum occurs when $p_4 = 0$; therefore

$$\mathcal{F}(T) \geq \frac{3}{2} + \frac{1}{6}\mathcal{E}(1)\,T$$

2) $T < \frac{1}{2}$

$$\mathcal{F}(T) \leq \max_{q_1, q_{-1}} \phi\left(T, q_{-1}, q_1, \tfrac{1}{2}, 0, \frac{1 + \frac{\mathcal{E}(1)}{3}}{2}\right)$$

$$= \max \left\{ 1 + (1-T)\frac{1}{2} + T\left[1 - q_1 - \frac{1 + \frac{\mathcal{E}(1)}{3}}{2}(1 - 2q_1)\right] + \frac{1}{3}\mathcal{E}(1)\;T\,(1 - q_1)\right\}$$

$$= \max\left\{\frac{3}{2} + \frac{\mathcal{E}(1)}{6}T\right\} = \frac{3}{2} + \frac{\mathcal{E}(1)}{6}T.$$

3) $T = \frac{1}{2}$. The result of 1) holds for any p_2, p_4, p_5.

In place of 2):

$$\mathcal{F}(\tfrac{1}{2}) \leq \max_{q_{-1}, q_1} \phi\left(\tfrac{1}{2}, q_{-1}, q_1, \frac{1 + \frac{\mathcal{E}(1)}{6}}{2}, \tfrac{1}{2}, \frac{1 + \frac{\mathcal{E}(1)}{6}}{2}\right)$$

$$= \max\left\{1 + \frac{1}{2}\left[1 - q_{-1} - \frac{1 + \frac{\mathcal{E}(1)}{6}}{2}(1 - 2q_{-1})\right] + \frac{1}{2}\left[1 - q_1 - \frac{1 + \frac{\mathcal{E}(1)}{6}}{2}(1 - 2q_1)\right] + \frac{1}{6}\mathcal{E}(1)\left[1 - \frac{q_1}{2} - \frac{q_{-1}}{2}\right]\right\}$$

$$= \frac{3}{2} + \frac{\mathcal{E}(1)}{6} \cdot \frac{1}{2}$$

<u>Theorem 3</u>. $\mathcal{E}(1) = \frac{5}{2}$ and the optimal strategies, when S = 1, are

$A = B = 0,\ C = 1,\ P_2 = P_3 = 0.$

<u>Proof</u>.

1) $$\mathcal{E}(1) \geq \min_{P_2, P_3} \left\{ 1 + 1 + \frac{1}{3}\mathcal{F}(1) + P_2 \frac{1}{2}\,[0] + P_3 \left[-\frac{1}{3}\mathcal{F}(1) + \frac{1}{3}\mathcal{E}(1) \right] \right\}.$$

As $\mathcal{F}(1) = \mathcal{F}(0) = \frac{3}{2}$

$$\mathcal{E}(1) \geq \min_{P_3} \left\{ \frac{5}{2} + \frac{P_3}{3} \left[\mathcal{E}(1) - \frac{3}{2} \right] \right\}.$$

Suppose now $\mathcal{E}(1) < \frac{3}{2}$. The minimizing value of P_3 is 1 and we have

$$\mathcal{E}(1) \geq \frac{5}{2} + \frac{1}{3}\mathcal{E}(1) - \frac{1}{2}$$

$$\frac{2}{3}\,\mathcal{E}(1) \geq 2,\quad \mathcal{E}(1) \geq 3,$$

a contradiction. Thus $\mathcal{E}(1) \geq \frac{3}{2}$ and the minimizing value of P_3 is 0.

$$\mathcal{E}(1) \geq \frac{5}{2}.$$

2) $$\mathcal{E}(1) \leq \max_{A,B,C} \left[1 + (A+2C)\frac{1}{2} + \frac{A+C}{3}\mathcal{F}\left(\frac{C}{A+C}\right) \right] = \max_{A,B,C} \beta(A,B,C).$$

We note $$\beta(0,0,1) = 1 + 1 + \frac{1}{3}\mathcal{F}(1) = \frac{5}{2}. \qquad (8)$$

Suppose that β is maximized by a set of values for which

$$\frac{C}{A+C} < \frac{1}{2}.$$

Then $C < \frac{1}{2}(A+C)$ or $C < A$. From $C \leq 1 - A < 1 - C$ we infer $C < \frac{1}{2}$.

For such A and C we can apply Theorem 2 and we find

$$\beta = 1 + \frac{1}{2} A + C + \frac{A+C}{3} \left(\frac{3}{2} + \frac{\mathcal{E}(1)}{6} \frac{C}{A+C} \right)$$

$$= 1 + A + C \left(\frac{3}{2} + \frac{\mathcal{E}(1)}{18} \right) \leq 1 + (1-C) + C \left(\frac{3}{2} + \frac{\mathcal{E}(1)}{6} \right)$$

$$= 2 + C \left(\frac{1}{2} + \frac{\mathcal{E}(1)}{18} \right) \leq 2 + \frac{1}{2} \left(\frac{1}{2} + \frac{3}{18} \right) = 2 \frac{1}{3} .$$

In virtue of (8), a maximum is not attained. Thus when it is attained

$$\frac{C}{A+C} \geq \frac{1}{2}$$

and $\mathcal{F}\left(\frac{C}{A+C}\right) = \frac{3}{2} + \frac{\mathcal{E}(1)}{6} \frac{A}{A+C}$. Thus

$$\beta = 1 + \frac{1}{2} A + C + \frac{A+C}{2} + \frac{\mathcal{E}(1)}{18} A$$

$$= 1 + \left(1 + \frac{\mathcal{E}(1)}{18}\right) A + \frac{3}{2} C$$

Now $1 + \frac{\mathcal{E}(1)}{18} \leq 1 + \frac{3}{18} < \frac{3}{2}$. Therefore the maximum occurs when C = 1, A = 0 and finally

$$\mathcal{E}(1) \leq 1 + \frac{3}{2} = \frac{5}{2} .$$

Now that we know $\mathcal{F}$ we return to (3) and (4) and put Ψ in a suitable form for computation.

$$\mathcal{E}(S) = \max_{\substack{A,B,C \\ C \leq S}} \ \min_{P_2, P_3} \Psi(A,B,C,P_2,P_3) \qquad (4)$$

with

$$\Psi = 1 + \frac{1}{2} A + C + \frac{A+C}{3}\left[\frac{3}{2} + \frac{5}{12}\frac{\min(A,C)}{A+C}\right] + P_2 \ \frac{1}{2}\left[B - A\right]$$

$$+ P_3\left[\text{———} + \frac{B+C}{3}\,\mathcal{E}\left(\frac{C}{B+C}\right)\right]$$

$$= 1 + A + \frac{3}{2} C + \frac{5}{36}\min(A,C) + P_2 \frac{1}{2}\left[B - A\right]$$

$$+ P_3\left[-\frac{A+C}{2} - \frac{5}{36}\min(A,C) + \frac{B+C}{3}\,\mathcal{E}\left(\frac{C}{B+C}\right)\right]. \qquad (9)$$

We will write (4) in the form

$$\mathcal{E}(S) = \max_{C \leq S} \ \mu(C) \qquad (10)$$

where

$$\mu(C) = \max_{A,B} \ \min_{P_2,P_3} \ \Psi.$$

We can eliminate B by replacing it by 1 - C - A. Further in Ψ we will replace the $\mathcal{E}$ term by its μ expression utilizing (10). Then

$$\mu(C) = 1 + 3/2\, C + \max_{\substack{A \\ 0 \leq A \leq 1-C}} \ \theta(A) \qquad (11)$$

where

$$\theta(A) = \min_{P_2,P_3}\left\{ A + \frac{5}{36}\min(A,C) + P_2\left[\frac{1-C}{2} - A\right]\right.$$

$$\left. + P_3\left[-\frac{A+C}{2} - \frac{5}{36}\min(A,C) + \frac{1-A}{3} \max_{r \leq \frac{C}{1-A}} \mu(r)\right]\right\}^{*} \qquad (12)$$

* It will turn out that μ is increasing. Thus the last factor of the last term can be replaced by $\mu(\frac{C}{1-A})$ and (10), by $E(S) = \mu(S)$. But we do not know this _a priori_. We regard C as fixed in θ and prefer not to write it $\theta(A,C)$.

In (12) we shall designate the coefficient of P_3 by λ. We shall aim at μ rather than E. Our precedure will ascertain successively the constants $C_1, \ldots, C_5$ with

$$0 < C_5 < \ldots \quad C_1 < 1.$$

We shall first find μ in the interval $[C_1, 1]$. The values there will be precisely those needed to know the μ term in λ when μ is calculated for C in the interval $[C_2, C_1]$. We proceed thus. When we know the values of μ in one interval, these values will suffice to determine λ in the next so that μ can be found there. The C_j come to light successively with the intervals.

A graphical diagram will facilitate matters.

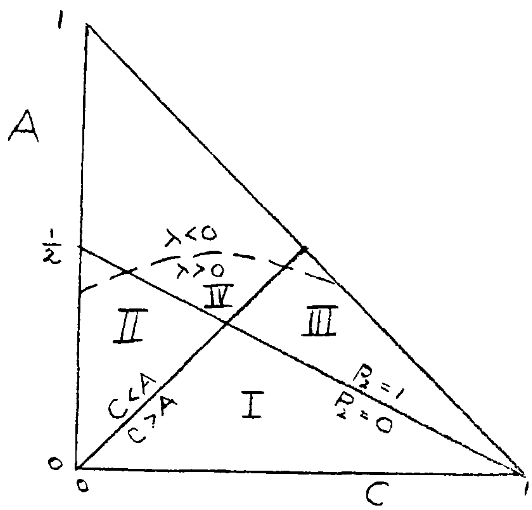

Fig. 3

In the C,A-plane we are concerned only with the region

$$0 \leq C \leq 1$$
$$0 \leq A \leq 1 - C$$

which is the triangle sketched in Fig. 3. Because min(A,C) appears in (12) we divide the triangle by the line where A = C. We also draw the line

$$A = \frac{1-C}{2}$$

where the coefficient of P_2 changes sign. Below this line this coefficient is positive and so the minimizing P_2 is 0; above the line it is 1.

Very important on the diagram is the curve where $\lambda = 0$. We do not know it yet, of course, but it is helpful to mark it tentatively; it is indicated in Fig. 3 as the dashed line. We shall discover subsequently that $\lambda > 0$ below it and $\lambda < 0$ above. Below it, then the minimizing $P_3 = 0$, and we can calculate θ there.*

Lemma 5. In the regions I, II, III of Fig. 3, θ is an increasing function of A. In IV, θ is constant and equals

$$\frac{1}{2} - \left(\frac{1}{2} - \frac{5}{36}\right) C. \qquad (13)$$

Proof:

In I, $\theta = A\left(1 + \frac{5}{36}\right)$

$$\text{In II, } \theta = A + \frac{5}{36} C \qquad (14)$$

$$\text{In III, } \theta = A + \frac{5}{36} A + \frac{1-C}{2} - A = \frac{5}{36} A + \frac{1-C}{2} \qquad (15)$$

In IV, $\theta = A + \frac{5}{36} C + \frac{1-C}{2} - A$

Thus we may expect to find no maximizing values of θ below the dashed line (or strictly when $\lambda > 0$) except in IV.

Let $\alpha(C)$ be such that when it replace A in λ, λ vanishes; in other words $A = \alpha(C)$ is the equation of the dashed curve.

Theorem 4. There is a number C_1 such that

$$C_1 = \frac{23}{35}$$

and when $C_1 \leq C \leq 1$,

$\lambda \geq 0$ for all A in $[0, 1-C]$

$\mu(C) = \left(\frac{3}{2} + \frac{5}{36}\right) + \left(1 - \frac{5}{36}\right) C.$

---

* If the reader objects to working subject to an unproved result, he can recast Lemma 5 in another form by adding the hypothesis that $P_3 \lambda = 0$.

The optimizing values are

$$P_2 = 1, \; P_3 = 0$$

$$A = 1 - C.$$

<u>Proof</u>: Let M denote the last term of (11), i.e.,

$$M = \max_{A} \Theta = \max_{A} \min_{P_2, P_3} \left\{ \quad \right\}.$$

Let $C > \frac{1}{2}$.

1) In the braces above put $P_2 = 1$, $P_3 = 0$.

$$M \leq \max_{A} \left\{ \frac{1-C}{2} + \frac{5}{36} \min (A, C) \right\}$$

Now $A \leq 1 - C \leq 1 - \frac{1}{2} = \frac{1}{2} \leq C$ and so

$$M \leq \max \left\{ \frac{1-C}{2} + \frac{5}{36} A \right\}.$$

The maximizing A is $1 - C$;

$$M \leq (1-C) \left(\frac{1}{2} + \frac{5}{36}\right).$$

2) We note, that if $A = 1 - C$,

$$\max_{r \leq \frac{C}{1-A}} \mu(r) = \max_{r \leq 1} \mu(r) = \mathcal{E}(1) = \frac{5}{2}. \tag{16}$$

Taking A as $1 - C$ (then, as above, $A \leq C$) we find

$$M \geq \min_{P_2, P_3} \left\{ (1 + \frac{5}{36})(1-C) - P_2 \left[\frac{1-C}{2}\right] + P_3 \left[-\frac{1}{2} - \frac{5}{36}(1-C) + \frac{C}{3} \cdot \frac{5}{2} \right] \right\}.$$

The non-negativity of the latter bracket (i.e., λ) is in turn equivalent to

$$-\frac{1}{2} - \frac{5}{36}(1-C) + \frac{5}{6} C \leq 0$$

$$\frac{-23 + 35C}{36} \geq 0$$

$$C \geq \frac{23}{35}. \quad (>\frac{1}{2})$$

When this inequality holds, the minimizing P_2 and P_3 are 1 and 0 and so

$$M \geq (\frac{1}{2} + \frac{5}{36})(1-C).$$

3) $\mu = 1 + \frac{3}{2} C + M = 1 + \frac{3}{2} C + (\frac{1}{2} + \frac{5}{36})(1-C).$

We now embark on our interval-by-interval calculation. As the full process is tedious and repititious we shall abandon a rigid format, compute one interval in full as a sample, make a few remarks, and state the result.

Now for $[C_2, C_1]$! We first suppose $\frac{1}{2} \leq C \leq C_1$ so that we may retain $A \leq C$. This turns out true for the maximizing A on the whole interval. We need not reinvestigate our $\alpha(C)$ under the premise $A > C$ as the term $\frac{5}{36} \min (A,C)$ is too small to cause a sign change.

The interval $[C_2, C_1]$ is selected so that the term $\max_{r \leq \frac{C}{1-A}} \mu(r)$ in λ can be evaluated from the μ values of $[C_1, 1]$. Here μ is increasing and λ becomes (see (12))

$$- \frac{A+C}{2} - \frac{5}{36} A + \frac{1-A}{3} \left[\frac{3}{2} + \frac{5}{36} + (1 - \frac{5}{36}) \frac{C}{1-A} \right] .$$

The coefficient of A is

$$-1 - \frac{10}{54}.$$

Comparing with (15) and referring to Fig. 3, we see that $\theta(A)$ is decreasing when we are above the dotted line. This means that the maximizing A is $\alpha(C)$.

We suppose that in $[C_2, C_1]$, $\alpha(C)$ has the form

$$a - bC$$

with a and b constants. The range of validity is to be those C for which

$$C_1 \leq \frac{C}{1-\alpha(C)} \leq 1$$

or

$$C_1 \leq \frac{C}{1-a+bC} \leq 1$$

or

$$C_2 = \frac{C_1(1-a)}{1-b\,C_1} \leq C \leq \frac{1-a}{1-b} = C_1' \qquad (17)$$

(this line numerically defining C_2 and C_1').

As $\alpha(C)$ annulls λ, we must have

$$- \frac{\alpha+C}{2} - \frac{5}{36} \alpha + \frac{1-\alpha}{3} \left[\frac{3}{2} + \frac{5}{36} + (1 - \frac{5}{36}) \frac{C}{1-\alpha} \right] = 0$$

or

$$\alpha(- \frac{1}{2} - \frac{5}{36} - \frac{1}{2} - \frac{5}{108}) + C\,(- \frac{1}{2} + \frac{1}{3} - \frac{5}{108}) + \frac{1}{2} + \frac{5}{108} = 0.$$

Solving for α, we find

$$a = \frac{59}{128}, \quad b = \frac{23}{128}.$$

Then from (17) we compute

$$C_2 = \frac{529}{1317}$$

and

$$C_1' = \frac{23}{35} = C_1 \text{ as we should expect.}$$

We also compute that

$$\alpha(C_2) = \frac{512}{1317} < \frac{529}{1317}$$

so that our statement that $\alpha(C) \leq C$ holds for the entire interval is verified.

We also find that

$$\alpha(C_1) = 1 - C_1$$

so the dashed curved meets the hypotenuse of the triangle (see Fig. 3) where $C = C_1$.

We can compute μ from (12) using for arguments:

$$A = \alpha(C),\ P_2 = 1,\ \lambda = 0.$$

It turns out that

$$\mu(C) = \left(\frac{3}{2} + \frac{5}{36}\ \frac{59}{128}\right) + \left(1 - \frac{5}{36}\ \frac{23}{128}\right) C.$$

We now have all the information about our interval except the optimizing P_3. To find this we write the quantity to be minimaxed from (12) (taking $P_2 = 1$):

$$A + \frac{5}{36} A + \left[\frac{1-C}{2} - A\right] + P_3 \left[-\frac{A+C}{2} - \frac{5}{36} A + \left(\frac{1}{2} + \frac{5}{108}\right)(1-A) + \left(\frac{1}{3} - \frac{5}{108}\right) C\right].$$

The coefficient of A is

$$\frac{5}{36} + P_3\left[-\frac{1}{2} - \frac{5}{36} - \frac{1}{2} - \frac{5}{108}\right]. \qquad (18)$$

If this coefficient were not zero the maximizing A would have to be one of its extreme values and not $\alpha(C)$. Thus equating (18) to 0 yields

$$P_3 = \frac{15}{128}.$$

<u>Remark</u>: Unlike α and μ, P_3 is discontinuous at C_1. One may ask what value it should have when $C = C_1$. A closer scrutiny of our analysis informs us that then all values of P_3 in the interval $\left[0, \frac{15}{128}\right]$ are optimizing.

The next interval is dealt with similarly. We assume a linear form for $\alpha(C)$. However the straight line which is its graph soon meets the line $A = C$ of Fig. 3. The intersection brings the interval to an end.

We are then finished as far as μ is concerned, for θ is constant in the region IV of Fig. 3 and above the dotted curve it decreases as a function of A. Its constant value in IV thus produces the optimum μ.* Clearly $P_2 = 1$, $P_3 = 0$ are optimal here, but we cannot know the best A without continuing the computation of α. Three more intervals and this is done.

The results are tabulated below. Fig. 4 is a plot of μ (or $\mathcal{E}$); Fig. 5 repeats Fig. 3 with the α curve in final form. Fig. 6 represents some of the less simple strategies. For the sake of completeness Figs. 7 and 8, which depict the second position, are included.

* This statement of course pends some simple assumptions, e.g., that the dotted curve does not enter region II (See Fig. 3), but these are easily disposed of.

Interval	$\mu(C)$ or $\mathcal{E}(C)$	$\alpha(C)$	P_3
1			
	$(\frac{3}{2} + \frac{5}{36}) + (1 - \frac{5}{36})C$	—	0
$C_1 = \frac{23}{35}$			
	$(\frac{3}{2} + \frac{5}{36}\ \frac{59}{128}) + (1 - \frac{5}{36}\ \frac{23}{128})C$	$\frac{59 - 23C}{128}$	$\frac{15}{128}$
$C_2 = \frac{529}{1317}$			
	$(\frac{3}{2} + \frac{5}{36}\ \frac{7207}{16039}) + (1 - \frac{5}{36}\ \frac{2419}{16039})C$	$\frac{7207-2419C}{16039}$	$\frac{1920}{16039}$
$C_3 = \frac{7207}{18458}$			
	$\frac{3}{2} + (1 + \frac{5}{36})C$ (for all intervals from C_3 to 0)	$\frac{7207-4339C}{14119}$	0 (for all intervals from C_3 to 0)
$C_4 = \frac{28566}{127339}$			
		$\frac{902141-51382C}{1768247}$	
$C_5 = \frac{19485}{895834}$			
		$\frac{1}{2} - \frac{7}{27}C$	
0			

We observe the $\mu(C)$ is increasing. From (10) we conclude

$$\mathcal{E}(S) = \mu(S). \tag{19}$$

We also infer the following important facet of the optimum strategies.

<u>Theorem 5</u>: The optimizing value of C is S.

This means that when E is two units from P he should always move away from P. Thus to express **E's** strategy we need only state what he should do when one space from P. We revert to the terminology if the proof of Lemma 1 and concentrate on Q_{10} (hereafter simply Q), the probability that E remains immobile when adjacent to P. Its value ascertained from (1) and (2) is unique:*

$$Q = \frac{A}{1-S}$$

and it is plotted in Fig. 6. The curve consists of segments of hyperbolas; the left part is abritrary as long as it stays in the (closed) shaded region.

It remains to say a few words about the original problem. Let E start his flight at an appreciable distance to, say, the right of P. We have seen that his best policy is to move away from E when separated by two spaces; then a _fortiori_ likewise when the separation is more than two spaces. Thus when the players are more than two spaces apart their tactics are naively direct; E flees and P pursues. This policy favors P's information. At any time the signal spots E at one of three points. If his next move is certified as being one jump to the right, P maintains his knowledge of one of three points. The next signal may coincide with these three points; if it does not, E is spotted to within one or two points. Thus, as the play progresses, this sifting continues and it is probable that E's position will soon be known to P.

Of course it is possible for E to deceive P by adopting less inflexible tactics, but Theorem 5 (and, if true, the inference drawn from it above) shows that it does not pay him to do so.

It appears that the direct phase of the game is quite simple. Clearly it must always lead either to capture, a first, or a second position. If the initial positions were widely separated, the value of S or T in the latter two cases will probably be 1; if not, it depends on the probabilities of P's location at starting.

---

* i.e., unique in terms of A. In Fig. 6 for certain C (or S in virtue of (19)), A make take any value which the ordinate of a point in the shaded region.

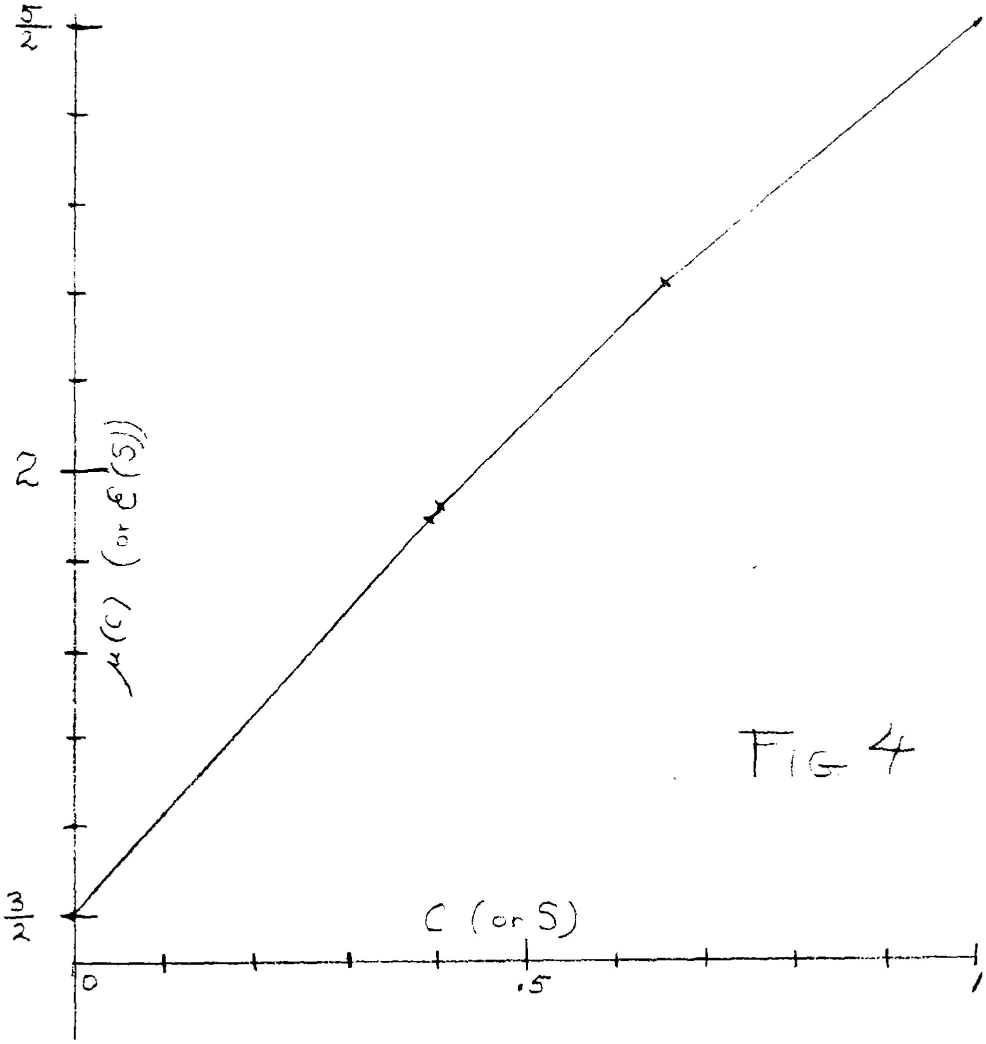

Fig 4

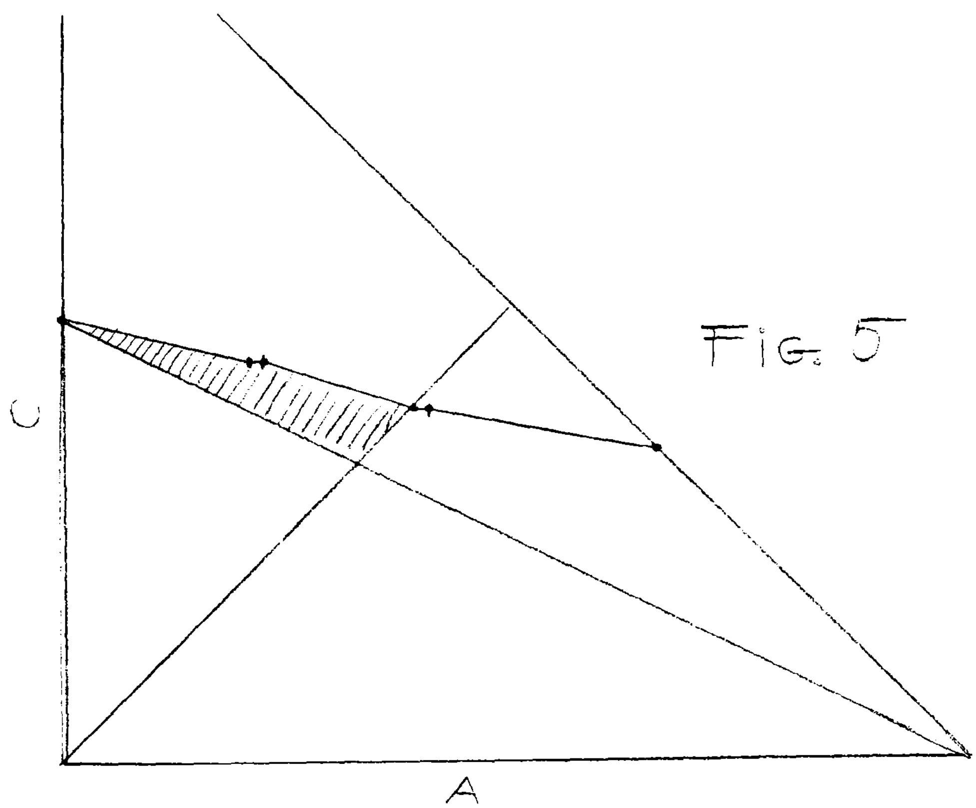

Fig. 5

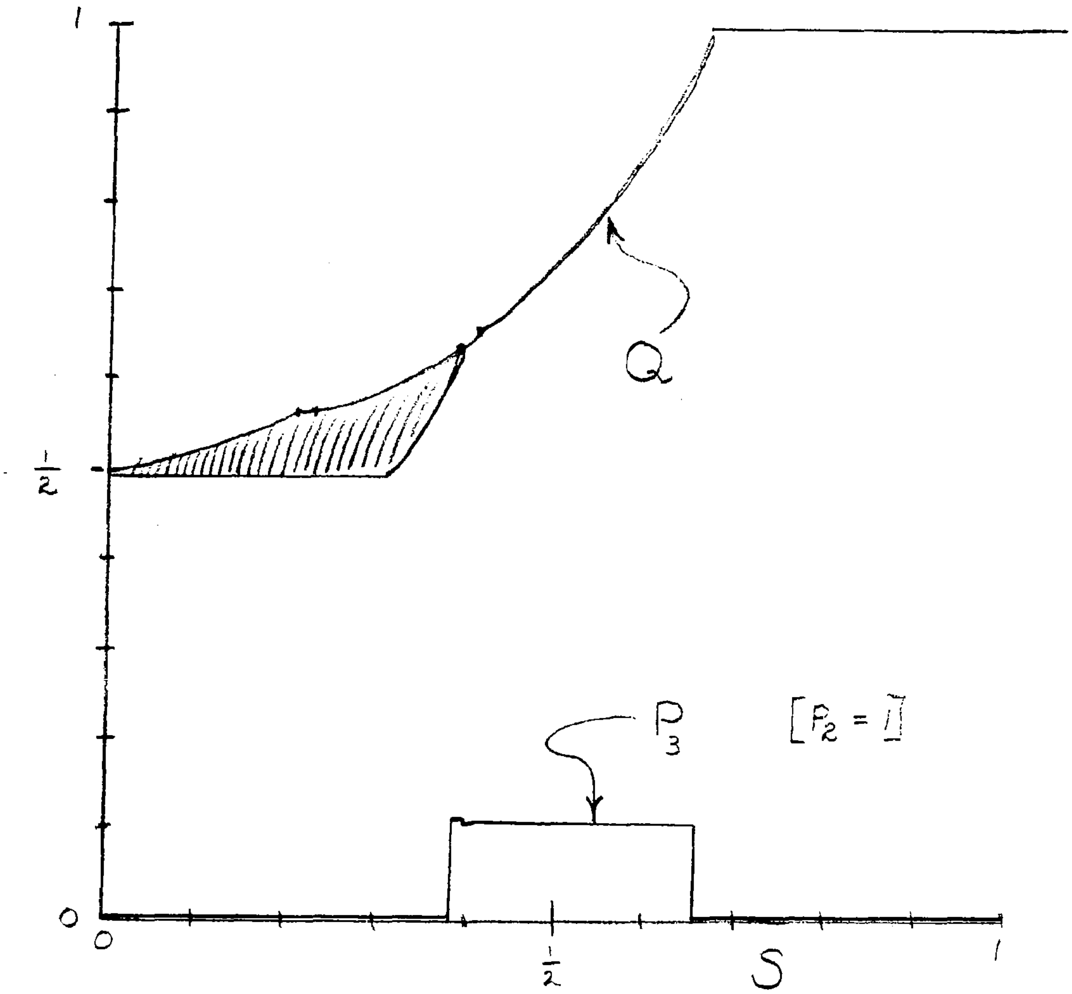

FIG. 6

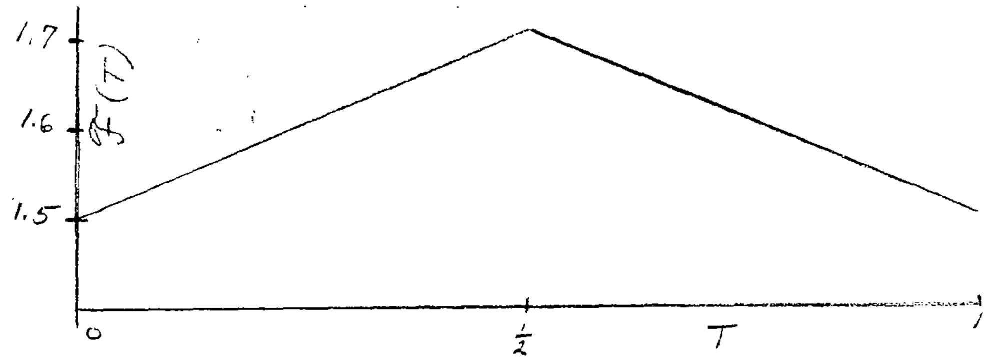

Fig. 7

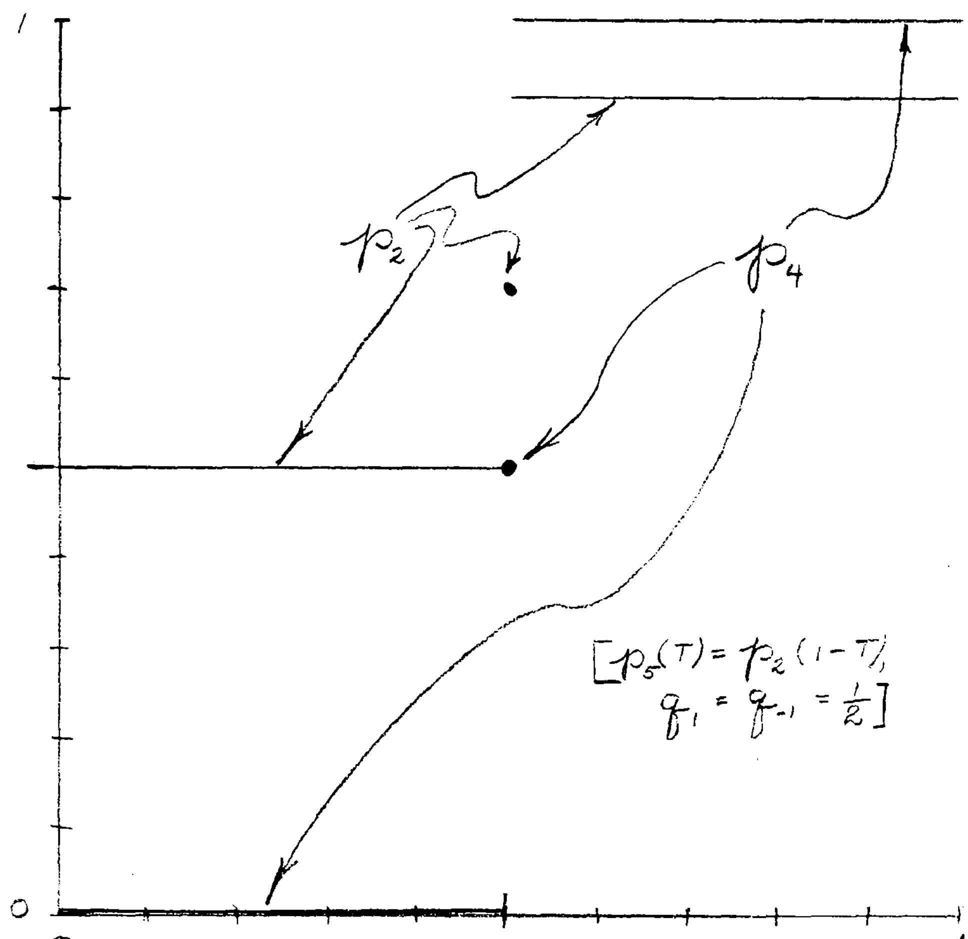

Fig 8